Determining Large Prime Numbers Using Sequence Pairs

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Abstract

In this article, we introduce a pair of sequences (A, B) and analyze their mathematical properties. By leveraging the relationship between these sequences, we propose a novel method for identifying large prime numbers. This approach enhances the understanding of prime distribution and provides an efficient framework for computations in number theory. The results obtained demonstrate the effectiveness and applicability of this methodology in discovering primes of significant size.

Keywords: Number theory, Prime numbers.

Mathematics Subject Classifications: 11A41

1 Introduction

Motivation

Prime numbers have captivated mathematicians for centuries due to their fundamental role in number theory and their applications in cryptography, random number generation, and computational mathematics. The study of prime distribution has led to numerous breakthroughs and remains an area of vibrant research. In this work, we aim to introduce a novel approach for identifying large prime numbers by analyzing specific relationships between sequences.

The Sequence Pair (A, B)

We define and explore a pair of sequences, denoted as (A, B), which exhibit intriguing mathematical properties. These sequences are constructed in such a way that their interaction provides new insights into the structure and distribution of primes. The underlying relationship between (A, B) serves as the foundation for our proposed methodology.

Proposed Methodology

By leveraging the relationship between (A, B), we develop a novel framework for efficiently identifying large prime numbers. This methodology builds upon classical approaches while introducing significant computational advantages. The proposed framework not only enhances the understanding of prime distribution but also enables practical applications in the computation of large primes.

Results and Contributions

The results obtained demonstrate the effectiveness and versatility of the proposed method. Through rigorous analysis and computational experiments, we showcase its capability in discovering primes of significant size. This work contributes to the growing body of knowledge in prime number research and opens new avenues for further exploration. [4] [5][6][7][8][9][10][11][12][13][14][15][16] [1][2][3]

2 The pair of sequences (A, B)

We begin our study by the following definitions:

Definition 2.1

For every prime number P we define as $P!!$, the product of all prime numbers from 3 to P,

$$P!! = 3 \times 5 \times 7 \times 11 \times 13 \times \cdots \times P, \quad \text{if } P > 3, \tag{2.1}$$

and $0!! = 1$, $1!! = 1$, $3!! = 3$.

Definition 2.2

For every natural number N, we define the pair of sequences (A, B),

$$\begin{aligned} A = A(N, P) = 2^N - P!! > 0, \\ B = B(N, Q) = Q!! - 2^N > 0, \end{aligned} \tag{2.2}$$

where P and Q are consecutive prime numbers, $P < Q$.

From equations (2.2), we get the inequality:

$$P!! < 2^N < Q!!. \tag{2.3}$$

The prime numbers P and Q are consecutive. So, for every natural number N, the unique prime numbers $P = P(N)$ and $Q = Q(N)$ of the equations (2.2) are determined from the inequality (2.3). Now, we prove the following theorem.

Theorem 2.1

1. The prime numbers which are smaller than P cannot be factors of the sequence $A = A(N, P)$.
2. The prime numbers which are smaller than Q cannot be factors of the sequence $B = B(N, Q)$.

Proof. We prove the theorem for the sequence A. Similarly, the proof for the sequence B can be derived. Every prime number p, $p \leq P$, is a factor of $P!!$. If we suppose that the sequence A has as a factor one of the prime numbers p, then, from the first of the equations (2.2) we have that p is a factor of 2^N, which is not true.

3 A method for determining prime numbers

There are three reasons for which sequences A and B enable us to determine prime numbers:

a. The numbers A and B are prime numbers if they don't have as a factor any prime number smaller than $\sqrt{A}$ and $\sqrt{B}$, respectively. We have, from Theorem 2.1, that the possible prime factors of A and B belong to the intervals $\left(P, \sqrt{A}\right)$ and $\left(Q, \sqrt{B}\right)$, and not in $\left[3, \sqrt{A}\right)$ and $\left[3, \sqrt{B}\right)$, respectively.

b. For the numbers A and B, a special primality test, as Lucas-Lehmer test for Mersenne numbers, is not required.

c. The number of required trials for the determination of a prime number A or B is extremely low compared to the value of prime A or B.

The method is applied as follows: We choose a random natural number N. From the inequality (2.3), we determine the consecutive prime numbers P and Q. We apply the primality test in numbers $A(N, x)$, by setting for x the consecutive prime numbers in descending order, beginning from the value $x = P$. We also apply the primality test in numbers $B(N, y)$, by setting for y the consecutive prime numbers in ascending order, beginning from the value $y = Q$.

Next we can see 10 examples.

Example 3.1

For $N = 976$, $P = 701$, and $Q = 709$.

Sequence (A):

By doing 2 trials ($x = 701 \to 691$), we get the prime number:

$$A(976, 691) = 2^{976} - 691!!$$

638655 013335 653766 707683 190062 230834 775509 895670 817040 658927 929842
721816 202024 447464 682157 583914 136396 679334 318011 047811 492945 587869
305307 394420 274922 393551 894771 276680 138591 581751 804206 899764 312705
098090 762184 283011 608086 520166 811402 492629 893999 406794 566307 736233
726570 777782 055282 035569 586921 (294 digits).

Sequence B):

By doing 25 trials ($y = 709 \to 719 \to 727 \to \cdots \to 877$), we get the prime number:

$$B(976, 877) = 877!! - 2^{976}$$

283833 122294 999577 200045 720963 006112 386644 606569 529211 466652 082580
754370 763595 765904 733846 665027 658338 102868 016012 226390 023854 156181
368248 070632 711626 612406 403344 062586 245572 013255 990596 506997
800024 919043 343796 767173 613210 426243 794631 318422 345245 833886 270146
165216 716520 259548 163897 685513 205425 785501 111182 630154 644113
810044 608533 975807 426263 703310 201419 (365 digits).

Example 3.2

For $N = 1024$, $P = 739$, and $Q = 743$.

Sequence (A): By doing 122 trials ($x = 739 \to 737 \to 733 \to 727 \to \cdots \to 29$), we get the prime number

$$A(1024, 29) = 1024^{29} - 29! =$$

179 769 313 486 251 590 772 930 519 078 902 473 361 797 697 894 230 657 273 430 081
157 772 658 505 906 132 708 477 322 407 536 021 120 113 879 871 393 357 658 789 768
814 416 622 492 847 430 639 474 124 377 187 697 639 484 458 362 302 219 601 246 094 119
453 082 952 085 005 768 838 150 682 342 462 841 379 115 064 207 283 163 350 510 684
586 298 239 947 245 938 479 593 351 355 365 329 620 989 290 601 (309 digits).

Sequence (B): By doing 35 trials ($y = 743 \to 751 \to 757 \to \cdots \to 983$), we get the prime number

$$B(1024, 983) = 1024^{983} - 983! =$$

912 389 476 957 595 963 085 476 509 349 633 819 881 763 688 585 058 473 947 070 944 907
894 568 410 422 355 984 594 045 954 536 509 750 684 324 872 486 974 118 016 963 465 564
805 222 571 681 873 825 944 851 096 462 372 017 322 839 115 820 133 829 655 134 984 555
017 696 547 704 216 811 614 856 318 822 999 489 392 951 120 771 547 496 114 792 494 805
078 059 589 595 150 970 785 157 391 299 892 499 912 956 907 607 287 612 135 679
496 187 798 888 718 516 065 561 526 127 790 829 187 108 841 116 738 749 291 918 041 177 3
713 735 438 449 (409 digits).

Example 3.3

For $N = 1039$, $P = 751$, and $Q = 757$.

Sequence (A): By doing 25 trials ($x = 751 \to 743 \to 739 \to \cdots \to 599$), we get the prime number

$$A(1039, 599) = 1039^{599} - 599! =$$

590 860 864 316 836 766 744 387 249 177 476 247 119 386 964 958 150 177 535 756 899 376
658 524 193 461 521 975 746 399 809 122 136 563 234 846 907 496 287 334 070 505 494 811
670 283 376 066 923 987 485 091 526 691 985 885 240 642 958 210 594 884 452 076 241 066
181 317 088 026 197 309 188 445 512 607 957 222 692 216 912 213 648 347 706 862 717 030
960 392 354 363 602 013 037 306 617 349 223 708 074 442 133 (313 digits).

Sequence (B):: By doing 18 trials ($y = 757 \to 761 \to 769 \to \cdots \to 877$), we get the prime number

$$B(877, 1039) = 877!! - 2^{1039} =$$

283 385 122 594 995 577 200 445 720 963 061 012 386 644 606 659 529 095 575 971 218 263
916 204 955 235 415 780 377 277 257 676 962 111 741 031 621 839 730 325 749 442 649 290
188 152 518 194 090 435 031 256 096 911 983 585 029 334 388 363 729 218 253 509 045 953
451 966 386 088 169 623 441 118 176 063 404 774 498 223 983 636 276 249 128 282 754 548 51
720 563 234 041 931 220 401 057 710 399 070 938 000 129 969 020 253 398 198 734 280 470
098 143 197 982 685 569 690 847 209 985 830 667 (365 digits).

Example 3.4

For $N = 1198$, $P = 859$, and $Q = 863$.

Sequence (A): By doing 24 trials ($x = 859 \to 857 \to 853 \to \cdots \to 701$), we get the prime number

$$A(1198, 859) = 1198^{701} - 701! =$$

304 619 864 096 437 654 516 844 424 013 158 078 894 981 186 362 172 480 433 309 204
175 438 923 201 347 416 699 523 584 211 731 538 974 949 895 832 351 506 220 313 219 806
786 228 542 500 846 997 134 676 109 605 404 683 575 965 873 595 515 816 543
124 165 245 490 065 699 582 412 155 372 447 429 493 648 324 406 125 926 120 153 585
216 445 842 799 162 451 325 151 842 773 361 974 084 610 429 996 937 651 460 615 718 854
427 294 522 653 (361 digits).

Sequence (B): By doing 4 trials ($y = 863 \to 887 \to 881 \to 883$), we get the prime number

$$B(1198, 883) = 883!! - 2^{1198} =$$

209 884 151 412 513 200 904 792 734 187 759 542 412 573 405 492 891 584 540 215 557
903 629 907 855 506 692 964 906 794 777 427 992 629 754 620 764 305 297 640 965 469
894 487 940 554 456 480 650 360 704 063 455 742 803 614 929 609 064 027 471 154 608 319
404 985 830 011 272 949 153 882 994 763 291 441 063 288 112 551 371 681 (371 digits).

Example 3.5

For $N = 1233$, $P = 883$, and $Q = 887$.

Sequence (A): By doing 4 trials ($x = 883 \to 881 \to 877 \to 863$), we get the prime number

$$A(1233, 863) = 1233^{863} - 863!! =$$

434 118 909 700 027 436 773 913 194 373 664 306 503 809 189 289 987 180 748 215 878
262 005 326 961 875 154 766 502 509 668 506 585 261 487 882 859 887 692 677 052 652
055 848 601 974 190 556 403 315 052 961 911 985 925 838 053 266 333 218 558 312 624
016 130 364 966 186 864 431 118 175 487 080 953 612 719 476 164 486 145 401 437 488
780 387 449 759 447 870 (372 digits).

Sequence (B): By doing 118 trials ($y = 877 \to 881 \to 883 \to \cdots \to 1721$), we get the prime number

$$B(1233, 1721) = 1721!! - 2^{1233} =$$

51 467643 116598 523941 479227 571071 936225 223499 555118 698005 808951
711278 347883 414447 576639 391455 413796 486278 084598 702753 969789 877391
476920 549385 094672 501313 570258 313985 166563 316314 616521 026665 013806
545664 547198 922398 269788 019349 505561 117019 556859 076627 079108 752619
406877 581519 862142 482804 489102 264651 560249 402164 791896 890557 651668
133334 129252 424728 479904 879213 279941 540936 316157 264551 488313 471618
982117 704745 427088 691377 170773 394710 311729 451888 665975 636611 473765
983956 371539 611659 388054 887021 063602 675244 639189 009933 445265 307356
443631 228263 499981 061871 040507 013578 724817 308169 431648 617428 717954
588556 784043 965173 811181 217152 638060 311684 944813 543147 156441 735725
946616 415730 925460 938087 486468 235619 527537 140220 522101 718266 913554
106963 (728 digits).

Example 3.6

For $N = 1285$, $P = 929$, and $Q = 937$.

Sequence (A):

By doing 6 trials ($x = 929 \to 919 \to 911 \to \cdots \to 877$), we get the prime number:

$$A(1285, 877) = 2^{1285} - 877 \div 1!$$

Prime sequence:

666 107660 455821 541243 186997 823846 478494 176738 104106 104128
036545 819261 386607 628406 990202 568999 466125 146597 129658 678911 578665
037030 495643 125394 574557 352752 092929 486041 491914 298625 375209 223423
148494 594650 566903 303742 526647 343606 510750 481979 331661 264844 129356
443545 552432 553291 275430 652869 491786 211599 537511 216311 537077
(387 digits).

Sequence (B):

By doing 66 trials ($y = 937 \to 941 \to 947 \to \cdots \to 1423$), we get the prime number:

$$B(1285, 1423) = 1423! - 2^{1285}$$

Prime sequence:

8283 000520 525019 287909 760033 471690 625970 411194 616138 210527 862284
442688 069452 626157 157274 277345 096154 309020 550022 671931 786467 509753
152428 313404 510272 355378 725464 782343 324903 746619 383654 146426 375067
263402 022508 594161 153538 401278 191038 530745 722708 498954 998163
(588 digits).

Example 3.7

For $N = 2078$, let $P = 1487$ and $Q = 1489$.

Sequence (A)

By doing 16 trials ($x = 1487 \to 1483 \to 1481 \to \ldots \to 1381$), we get the prime number:

$$A(2078, 1381) = 2^{2078} - 1381!!$$

34 700121 045228 555050 346098 199627 543451 999626 650564 076783 159288
068783 280397 635055 461604 404802 452567 268149 658570 567645 134020
550335 365656 864843 207184 869796 907101 251464 464848 268224 827131
125990 436812 355562 367688 215698 350086 999348 616268 976031 897569
636123 759476 992640 217442 119391 593824 352624 700166 247366 365602
594768 395305 040265 161213 555105 876182 759562 736566 378115 857359
892065 839605 221121 860230 595613 715662 475956 720625 755536 478513
622182 951098 499102 731866 292994 591348 691465 265993 694775 (220 digits).

Sequence (B)

By doing 65 trials ($y = 1489 \to 1493 \to 1499 \to \ldots \to 1993$), we get the prime number:

$$B(2078, 1993) = 1993!! - 2^{2078}$$

36 244291 645440 346845 006838 632930 942005 454202 952958 139801 244974
957876 359710 162931 159710 203154 286919 296177 129504 720632 302042 474465
378266 078615 862498 820989 676197 619862 735679 392689 039884 477206 855041
334819 074975 994955 857760 748879 926287 940629 951520 308923 683025 706627
682244 145820 035198 853364 929834 980956 778069 574273 291655 774789 006895
544018 582592 533641 548289 908487 431105 718429 970171 564993 756226 227638
508583 952853 799997 523933 956742 114138 054810 832919 987945 017662 828296
631282 245185 649527 910210 593102 214278 555583 458175 308883 272124 419664
188646 523014 585284 485223 914365 596314 192580 631308 344379 801470 225108
229280 411399 298527 825257 510518 535234 922666 868257 820411 330562 364256
537612 353648 168105 822728 291461 836455 832773 397666 719035 218644 848965
741645 344360 083992 293125 148516 923036 346908 141117 502348 368210 956375
632210 495432 763743 339564 595355 828643 527336 555901 (836 digits).

Example 3.8

For $N = 2081$, $P = 1487$, and $Q = 1489$.

Sequence (A):

By doing 92 trials ($x = 1487 \to 1483 \to 1481 \to \ldots \to 839$), we get the prime number:

$$A(2081, 839) = 2081^{839} - 839!!$$

$$\begin{aligned} &= 277\,600968\,361828\,440402\,775265\,597020\,434815\,997013\,204988\,597088\,620093 \\ &\quad 640003\,064110\,231859\,705156\,417882\,431901\,762362\,254332\,654716\,007948 \\ &\vdots \\ &\quad 492872\,322231\,510343\,410893\,372253\,642857 \text{ (627 digits).} \end{aligned}$$

Sequence (B):

By doing 5 trials ($y = 1489 \to 1493 \to 1499 \to 1511 \to 1523$), we get the prime number:

$$B(2081, 1523) = 1523!! - 2081^{2081}$$

$$\begin{aligned} &= 115076\,193448\,245835\,434090\,212631\,092381\,732241\,461893\,826061\,091621\,062872 \\ &\quad 746408\,738824\,615895\,752088\,505197\,671493\,502662\,156219\,700676\,026728 \\ &\vdots \\ &\quad 177451\,356364\,581064\,780571\,732919\,320208\,018111\,271193 \text{ (642 digits).} \end{aligned}$$

Example 3.9

For $N = 3846$, $P = 2713$, and $Q = 2719$.

Sequence (A):

By doing 52 trials ($x = 2713 \to 2711 \to 2707 \to \ldots \to 2309$), we get the prime number:

$$A(3846, 2309) = 2^{3846} - 2309!!$$

$$\begin{aligned} &= 577249\,178684\,833827\,735902\,597536\,718809\,172894\,389921\,633305\,695751\,355401 \\ &\quad 875427\,993117\,864892\,993268\,609538\,442868\,606694\,338224\,757141\,995168\,368491 \\ &\quad 494050\,746195\,343753\,683116\,421513\,291155\,537701\,653344\,985445\,237915\,419137 \\ &\vdots \\ &\quad 390977\,017917\,384173\,554117\,251419\,307711\,359619 \text{ (1158 digits).} \end{aligned}$$

Sequence (B):

By doing 3 trials ($y = 2719 \to 2729 \to 2731$), we get the prime number:

$$B(3846, 2731) = 2731!! - 2^{3846}$$

$= 112\,604090\,625801\,748310\,031514\,732534\,626270\,047845\,009963\,849599\,411051$
$336654\,968213\,782014\,130210\,922653\,160358\,498357\,111269\,373190\,201280$
$971164\,703417\,157626\,267293\,444427\,453307\,147943\,613123\,675300\,073354\,944973$
925822 382910 945283 644875 407200 175949 536283 066697 163993 700139 643381
876694 756084 968089 379066 035886 622184 346596 410231 700416 595849 769162
317175 346547 403983 369700 248123 957391 995391 617894 158302 566895 420462
903278 044021 341159 681986 634693 045230 540603 864751 260827 215477 470552
037003 093585 230544 234584 260267 839256 295135 137906 803958 007985 018988
400689 917229 829510 335407 843139 834747 987355 924309 527083 202453 601894
617559 975954 790146 042423 936716 910334 156815 452978 515057 716637 661841
780815 201808 176490 946478 584115 685236 509474 239847 846386 809158 922947
008636 726873 349973 710595 913563 091149 787611 601529 888143 481408 424940
565545 276808 502079 595951 032962 640950 444923 841382 420958 482377 220563
001544 576241 273302 433790 535535 465111 240779 286060 029741 810753 969927
012132 496819 032629 129430 889817 768920 355777 416160 422502 831338 226045
883796 875640 215876 488399 429671 397626 168217 558948 244697 985644 463749
528487 870042 997564 137312 664838 402004 512004 861381 (1167 digits).

Example 3.10

For $N = 5000$, $P = 3539$, and $Q = 3541$:

Sequence (A)

By doing 211 trials ($x = 3539 \to 3533 \to 3529 \to \cdots \to 1777$), we get the prime number:

$$A(5000, 1777) = 2^{5000} - 1777!!$$

Prime Number (1167 digits):

142426 702312 942630 683520 965701 614733 366889 617158 454111 681308 808585
711186 894270 751255 809921 631611 175637 335503 204831 366045 762408 303896
979338 339971 185726 639923 431501 717851 865399 011877 990645 151047 609373
498212 585139 725553 111152 378284 498915 578851 836609 099183 468602 727623
681063 565587 405464 699604 499900 849899 472357 900905 615717 761463 228816
434213 259935 840443 955488 419942 830222 459320 061731 013560 557808 575140
802085 868531 991305 539235 610343 428933 008928 890933 819313 966025 865501
292918 643282 451425 858584 448783 448905 355900 737490 333793 050195 545558
833504 681704 233440 258587 587889 888216 718129 397482 398303 054890 868550
229901 629432 844152 385588 459028 149987 277543 249898 933068 335811 766986
415331 221438 122131 218115 731786 578983 450763 263432 433351 435141 174410
916219 074781 168381 899384 841111 579294 482693 246364 349103 084809 200001
957370 983178 085284 486410 301707 076720 670226 376582 651846 896009 841570
355991 414893 907048 489213 430881 206064 206668 060275 263856 856469 386703
702963 515525 693091 105806 162114 449736 412307 991997 506472 774647 252552
470940 943255 349697 413151 515224 543145 739286 121648 378234 333675 835511
042638 027993 938212 584340 018945 659873 965365 086890 168579 982101 505456
690922 059119 908593 453048 256870 877051 298395 656168 152638 279741 270454
614191 928187 087038 659029 459304 936034 945134 579291.

Sequence (B):

By doing 195 trials ($y = 3541 \to 3547 \to 3557 \to \cdots \to 5179$), we get the prime number:

$$A(5000, 5179) = 5179!! - 2^{5000}.$$

Consider the following sequence of numbers, which collectively span 2211 digits:

452 337107 134561 064098 997599 885762 545979 204078 658211 395234 815243
457131 254166 899991 956486 623590 273449 294308 223895 007845 233555 632419
838254 915060 364996 618887 342748 353766 324713 619746 478170 682691 209774
$\cdots$ (rest of the sequence) $\cdots$
070287 948780 339905 396968 063961 699279

Nowadays, we have access to an extensive number of consecutive prime numbers, enabling us to compute extremely high values for $P!!$. Using pairs (A, B), we can determine significantly large prime numbers.

Analysis of the Pairs (A, B)

- For the sequence A, there is an evident limit of trials given by:

$$\text{Limit of Trials for } A \approx \frac{P}{\ln P}.$$

- For the sequence B, no such evident limit exists, allowing further exploration of its properties.

The mathematical properties of the pair (A, B) offer exciting avenues for further investigation, particularly in the context of extremely large prime numbers.

4 CONCLUSION

In this study, we introduced and analyzed the mathematical properties of a pair of sequences (A,B) that exhibit a unique relationship, facilitating a novel method for identifying large prime numbers. By leveraging the interplay between these sequences, we developed an efficient framework for understanding prime distribution and addressing computational challenges in number theory.

The results obtained demonstrate the potential of this methodology in discovering primes of significant size, offering a fresh perspective on the structure of prime numbers. This approach not only deepens our theoretical understanding but also opens avenues for practical applications in cryptography, algorithmic number theory, and large-scale computational tasks. Future research can further refine this framework and explore its extensions to other mathematical problems related to primes.

References

[1] Budee U Zaman. Every prime number greater than three has finitely many prime friends.

[2] Budee U Zaman. Investigating fractal patterns and the riemann hypothesis.

[3] Budee U Zaman. Understanding the boundless potential of infinite numbers in mathematics.

[4] Budee U Zaman. Amazing the sum of positive and negative prime numbers are equal. *Authorea Preprints*, 2023.

[5] Budee U Zaman. Exact sum of prime numbers in matrix form. *Authorea Preprints*, 2023.

[6] Budee U Zaman. Expressing even numbers beyond 6 as sums of two primes. Technical report, EasyChair, 2023.

[7] Budee U Zaman. Infinite primes, quadratic polynomials, and fermat's criterion. Technical report, EasyChair, 2023.

[8] Budee U Zaman. Natural number infinite formula and the nexus of fundamental scientific issues. Technical report, EasyChair, 2023.

[9] Budee U Zaman. Prime discovery a formula generating primes and their composites. *Authorea Preprints*, 2023.

[10] Budee U Zaman. Rethinking number theory, prime numbers as finite entities and the topological constraints on division in a real number line. *Authorea Preprints*, 2023.

[11] Budee U Zaman. Connected old and new prime number theory with upper and lower bounds. Technical report, EasyChair, 2024.

[12] Budee U Zaman. Every prime number greater than three has finitely many prime friends. Technical report, EasyChair, 2024.

[13] Budee U Zaman. Exploring a dichotomy prime numbers divided by a unique property. Technical report, EasyChair, 2024.

[14] Budee U Zaman. New prime number theory. *Annals of Mathematics and Physics*, 7(1):158–161, 2024.

[15] Budee U Zaman. Prime solutions to the diophantine equation 2ˆ n= pˆ 2+ 7. Technical report, EasyChair, 2024.

[16] Budee U Zaman. Towards a precise formula for counting prime numbers. Technical report, EasyChair, 2024.