Mathematical Modeling and Algorithms for Studying Vibrations of a Beam with Built in Ends at Different Properties of the Material

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***Abstract*— The paper deals with mathematical modeling and algorithms for studying vibrations of a beam with built-in ends, with regard to rheological properties of the beam material. By the method of Laplace integral transform the solution of the integro-differential equation of a beam is constructed in the form of a series, whose first term coincides with the solution of this equation obtained by the averaging method. For a specific Abel kernel, algorithm and software for the numerical analysis of the stated problem was developed. It was obtained that the amplitudes of vibrations decrease exponentially over time and taking into account of the subsequent terms of the series of solutions enable to determine a more exact solution to the problem.**

Keywords— image, original, kernel, integro-differential, beam, amplitude, exponential law.

I. Introduction

Problems of studying vibrations of these or other mechanical systems have numerous applications and are one of the subjects of theoretical and applied studies.

II. Problem Statement

Assume that the beam is homogeneous and isotropic of finite length equal to $x = \ell$, and the symmetry axes of the beam coincides with the axis OX in its equilibrium state. Let the beam with built-in ends beginning with moment of time $t = 0$ perform tranverse vibratious.

The mathematical problem of forced transverse vibrations of the visco-elastic beam may be described by an integro-differential equation [3] of the form:

$$\frac{\partial^4 u(x,t)}{\partial x^4} + \frac{\partial^2 u(x,t)}{\partial t^2} - \int_0^t K(t-\tau)\frac{\partial^4 u(x,t)}{\partial x^4} d\tau = f(x,t) \qquad (1)$$

$$0 < x < 1; \; 0 < t < T.$$

where $f(x,t)$ is a given function, $u(x,t)$ is a lateral displacement, $k(t-\tau)$ is a positive kernel and contains it its representation a small parameter.

We accept the initial conditions as follows:

$$u(x,t) = u_0; \; u_t'(x,t) = v_0 \quad \text{for} \quad t = 0 \qquad (2)$$

The boundary conditions are determined in the form:

$$u(x,t) = 0; \; \frac{\partial u(x,t)}{\partial x} = 0 \quad \text{for} \quad x = 0 \text{ и } x = 1 \qquad (3)$$

Thus, the problem is reduced to solving equation (1) under initial-boundary conditions (2)-(3).

Problem solution. Applying the Bubnov-Galerkin procedure to the equation (1), we obtain the following differential equation [4]

$$X^{IV}(x) - \lambda^4 x(x) = 0 \qquad (4)$$

$$X(0) = 0; X''(0) = 0; \; X(\ell) = 0; \; X''(\ell) = 0 \qquad (5)$$

Here the problem (4)-(5) has eigen-values in the form $\lambda_k = \frac{\pi k}{\ell}$, $k = 1,2,3,...$ and integro-differential equations

$$T''(t) + T_k(t) - \varepsilon\int_0^t k(t-\tau)T_k(\tau)d\tau = \lambda_k^2 q_k(t) \qquad (6)$$

with initial conditions

$$T_k(t) = u_{0k}; \; T_k'(t) = v_{0k} \quad \text{for} \quad t = 0 \qquad (7)$$

Here $\varepsilon > 0$ is some small parameter.

$$q_k(t) = \frac{1}{\lambda_k^2}\int_0^\ell q(x,t)x_k(x)dx \qquad (8)$$

Solving the equation (4)-(5), we find the eigen – functions

$$x_n(k) = \frac{\sin\lambda_k \ell}{1+\cos r_k \ell}(\cos\lambda_k x - ch\lambda_k x) + sh\lambda_k x - \sin\lambda_k x \qquad (9)$$

To determine amplitude functions, it is required to find the solutions of the integro-differential equation (6) under initial conditions (7), that is one of the difficulties when solving this equation [4].

Applying the Laplace transform in time t to the equation (6) allowing for condition (7) we get:

$$\bar{T}(p)=\frac{su_{ok}+v_{ok}}{s^2+\lambda_k^2-\varepsilon\lambda_k^2\bar{k}(p)}+\frac{\bar{q}_k(s)}{s^2+\lambda_k^2-\varepsilon\lambda_k^2\bar{k}(p)} \qquad (10)$$

where s is a Laplace transform, $\bar{T}(s)$ and $\bar{q}(s)$ are Laplace images of the functions of the same name $T(t)$ and $q_k(t)$.

We represent the denominator of equation (10) in the form:

$$\frac{1}{s^2+\lambda_k^2-\varepsilon\lambda\bar{k}(s)}=\frac{1}{s^2+\lambda_k^2}\sum_{n=0}^{\infty}\left(\frac{\varepsilon\lambda_k^2\bar{k}(\rho)}{s^2+\lambda_k^2}\right)^n$$

Then the solution of (10) takes the form:

$$\bar{T}(s)=\frac{su_{0k}+v_{0k}}{s^2+\lambda_k^2}\sum_{n=0}^{\infty}\left(\frac{\varepsilon\lambda_k^2\bar{k}(\rho)}{s^2+\lambda_k^2}\right)^n+$$
$$+\frac{\bar{q}_k(s)}{s^2+\lambda^2}\sum_{n=0}^{\infty}\left(\frac{\varepsilon\lambda_k^2\bar{k}(s)}{s^2+\lambda_k^2}\right)^n. \qquad (11)$$

Due to the inequality

$$0\le\varepsilon\int_0^t k(\tau)d\tau<<1,\ \varepsilon k(t)\ge 0$$

series (11) is a convergent series. By means of convolution of functions we calculate Laplace inverse transform and omit the indices for simplicity of notations

$$L^{-1}\left\{\frac{\varepsilon\lambda^2\bar{k}(s)}{s^2+\lambda^2}\right\}=\frac{\varepsilon\lambda^2k_s-\varepsilon\lambda sk_s-\varepsilon\lambda(s^2+\lambda^2)\bar{z}(s)}{\lambda^2+s^2}$$

Taking this formula into account in (11) and summing the series, we obtain:

$$\bar{T}(s)=\frac{su_0+v_0}{\bar{a}(s)-\varepsilon\lambda^2 d}+\frac{\bar{q}(s)}{\bar{a}(s)-\varepsilon\lambda^2 d} \qquad (12)$$

where

$$K_c=\int_0^\infty K(\tau)\cos\lambda\tau d\tau;\quad K_s=\int_0^\infty K(\tau)\sin\lambda\tau d\tau$$

$$z(t)=\int_t^\infty K(\tau)\sin\lambda(t-\tau)d\tau$$

$$\bar{a}(\rho)=(s+\frac{1}{2}\varepsilon\lambda k_s)^2+\lambda^2(1-\frac{1}{2}\varepsilon k_c)^2$$

$$d=\frac{1}{4}(k_c^2+k_s^2)$$

It is clear that here $\left|\frac{\varepsilon\lambda^2 d}{\bar{a}(s)}\right|<1$. Then expanding in series, the expression (12), we obtain:

$$\bar{T}(s)=\frac{su_0+v_0}{\bar{a}(s)}[1+\varepsilon\lambda^2\frac{d}{\bar{a}(s)}+\varepsilon^2\lambda^4\frac{d^2}{\bar{a}^2(s)}+...+]=$$
$$=\frac{\bar{q}(s)}{\bar{a}(s)}[1+\varepsilon\lambda^2\frac{d}{\bar{a}(s)}+\varepsilon^2\lambda^4\frac{d^4}{a^4(s)}+...+] \qquad (13)$$

Here, the first addend describes free vibration, the second addend corresponds to the forced vibration that we denote by $T^c(t)$ and $T^b(t)$, respectively.

The originals of these functions are determined successively as follows:

For the first addend we have:

$$T_1^c(t)=e^{-\frac{1}{2}\varepsilon\lambda k_s t}\left[u_0\cos\lambda(1-\frac{1}{2}\varepsilon k_c)t+\right.$$
$$\left.+\frac{v_0-\frac{1}{2}\varepsilon\lambda k_s}{\lambda(1-\frac{1}{2}\varepsilon k_c)}\sin\lambda(1-\frac{1}{2}\varepsilon k_c)t\right]$$
$$T_2^c(t)=\varepsilon\lambda^2T_1^c(t)*g(t)=\varepsilon\lambda^2\int_0^t g(t-\tau)T_1^c(\tau)d\tau \qquad (14)$$

....

$$T_n^c(t)=\varepsilon\lambda^2T_{n-1}^c(t)*g(t)=\varepsilon\lambda^2\int_0^t g(t-\tau)T_{n-1}^c(\tau)d\tau$$

For the second addend we similarly determine:

$$T_1^b(t)=\frac{1}{\lambda(1-\frac{\lambda\varepsilon}{2}k_c)}=\int_0^t q(t-\tau)g(\tau)d\tau=$$
$$=\frac{1}{\lambda(1-\frac{\varepsilon k_c}{2})}\int_0^t q(t-\tau)e^{-\frac{1}{2}\varepsilon\lambda k_s\tau}\sin\lambda(1-\frac{1}{2}\varepsilon k_s)\tau d\tau$$
$$T_2^b(t)=\varepsilon\lambda^2T_1^b(t)*g(t)=\varepsilon\lambda^2\int_0^t T_1(t-\tau)g(\tau)d\tau \qquad (15)$$

....

$$T_n^b(t)=\varepsilon\lambda^2T_1^b(t)*g(t)=\varepsilon\lambda^2\int_0^t T_{n-1}(t-\tau)g(\tau)d\tau$$

where the function $g(t)$ is determined as follows

$$g(t)=L^{-1}\left[\frac{d}{\overline{a}(s)}\right]=\frac{d}{\lambda(1-\frac{1}{2}\varepsilon k_c)}e^{-\frac{1}{2}\varepsilon\lambda k_s t}\sin\lambda(1-\frac{1}{2}\varepsilon k_c)t \tag{16}$$

where L^{-1} is an operator of Laplace inverse transform?

Then the solution of integro-differential equation (6) is determined as follows:

$$T_n(t)=\sum_{n=1}^{\infty}[T_n^c(t)+T_n^b(t)] \tag{17}$$

From this equation for the first approximation of solutions we obtain:

$$\begin{aligned} T_1(t) &= T_1^c(t)+T_1^b(t)= \\ &= Me^{-\frac{1}{2}\varepsilon\lambda k_s t}\sin[\lambda(1-\frac{1}{2}\varepsilon k_c)t+\theta]+ \\ &+N\int_0^t q(t-\tau)e^{-\frac{1}{2}\varepsilon\lambda k_s \tau}\sin\lambda(1-\frac{1}{2}\varepsilon k_c)\tau d\tau \end{aligned} \tag{18}$$

where $M=\sqrt{\frac{(v_0-\frac{1}{2}\varepsilon\lambda k_s)^2}{N^2}+u_0^2}$; $\theta=\frac{u_0 N}{v_0-\frac{1}{2}\varepsilon\lambda k_s}$;

$N=\frac{1}{\lambda(1-\frac{1}{2}\varepsilon k_c)}$.

Note that (17) coincides with the solution of the equation (6) obtained by the well-known averaging method 1,2,3. Taking into account subsequent approximations allow to obtain a more exact solution of equation (6).

III. CONCLUSIONS

The obtained solution was studied by the numerical calculations for Abel's specific kernel, it is shown that for small values of frequency, the influence of the subsequent term of the series on the solution is negligible, and with increasing the following it increases and amplitude of all terms exponentially decrese.

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