The 4-Color Theorem is Proved by Hand

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June 12, 2021

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***Abstract*—We prove the four color theorem (briefly 4CT)by a new way, which is absolutely different from ones by A.B. Kempe in 1879 and P. Tait in 1880 as well as the computer-aided proofs by K. Appel and W. Haken in 1976 and by N. Robertson etc. in 1995.With a tier graph of the tier number being the least and two definitions: one is a vertex adjacent closed subgraph corresponding to v_i , another is the good independent sets; three conditions to get the first good independent set r_1 from any planar graph G had been found, by which V(G) can be partitioned into 4 independent sets.**

Finally, we show in detail the entire procedure to prove the 4CT by a example.

Keywords—outer planar graph, vertex adjacent closed subgraph, hub, good independent set, tier graph

I. INTRODUCTION

The four color problem first appeared in a letter of October 23, 1852 to Sir William Hamilton from Augustus de Morgan, which was asked to him by his student Frederick Guthrie who later attributed it to his brother Francis Guthrie. After the announcement of this problem to the London Mathematical Society by Arthur Cayley in 1878, within a year its solution was proposed by A.B. Kempe [9]. After 11 years this publication P. J. Heawood published its refutation [7]. Another proof by P.G. Tait [13]in 1880 again was negated by W.T. Tutte [14].

The four color problem in graph theory has stood out as unscalable peak for a century or more[11]. Until 1976 using A.B. Kempe's idea, K. Appel and W. Haken proposed a computer-aided proof of the 4CT [1,2,3], but it is too long and too complex to be tested by hand. In 1995 N. Robertson, D. Sanders P. Seymour and R. Thomas, still using A.B. Kempe's idea, gave another 4CT computer-aided proof [12], but simpler than Appel and Haken's in several respects, and easy to be tested by hand, so the 4CT is established.

Hereafter, some scholars consider that to prove the 4CT by hand is inadvisable[16] and impossible.

Here is a query that the 4 color problem belong to NP-c? Most books[15,16] point definitely out that in graph theory the vertex k-colorable problems, $k \geqq 3$, belong to NP-c problems. Expressly, k=3, it is seemingly simple tractable, in fact, it is heartbreaking one of NP-c problems!

The 4-color problem of a planar graph is one case of the k-colorable problems, so it should be one of the NP-c problems.

Now that, for the 4CT they got a good algorithm ($O(n^2)$)[1,2,3,12], then that P=NP should be gotten. By results by Cook[5] and Karp[8] all NP-c problems should have good algorithm. However, why no one of thousands NP-c problems is resolved for 40 and more years from 1976 to today?!

Studied their way proving the 4CT,we think that their way is an optimized enumeration. By which one can not deal with other NP-c problems.

We think that the primary way to prove 4CT is to partition vertices of a planar graph into 4 independent sets. But the problem finding independent sets is one of NP-c problems[6,15,16]in graph theory. And if found a good method that partition vertices of a 3-colorable graph into 3 independent sets, then it is established that P=NP[16].

For 40 years or more working we have found necessary and sufficient conditions finding the first good independent set, by which V(G) of a planar graph can be partitioned into 4 independent sets by only one time operation.

II. THE OUTER PLANAR GRAPH AND RELATIVE THEOREM

Let G be a simple maximal planar graph. It can be drawn in a plane without edge-crossing, and divides the plane into faces; all vertices and edges of G all are on the boundaries of faces, that is, there are no vertex and no edge in either the

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interior or exterior of any face. It has exactly one unbounded face, called the outer face. If all vertices of a planar graph are on the boundary of a face, the planar graph is called the outer planar graph and denoted by ϖ.

The undefined terms and symbols used in this paper can be found in [4].

Theorm1 An outer planar graph ϖ is 3-colorable.

Proof. By induction on the vertex number n on ϖ:

1.When $n \leq 3$ the theorem is immediate.

2.Suppose that the theorem holds when the vertex number of ϖ is fewer than n；

3. let the vertex number of ϖ be n. From such an edge $e_{i,j} = (v_i,v_j)$ whose degree of endpoints v_i and v_j are more than two (If not, G is a cycle, and is 3-colorable),split the ϖ into two ϖ_1 and ϖ_2, so the vertex number of each of both is fewer than n. So both ϖ_1 and ϖ_2 are 3-colorable by the second hypothesis. With 3-coloring the ϖ_1 and ϖ_2, the two pairs same name vertices (v_i and v_j in ϖ_1 and ϖ_2 respectively)should be colored with the same colors, respectively, which is easy to done, then merging these two ϖ_1 and ϖ_2 forms the outer planar graph ϖ, which had been colored in 3 colors.∎

III. DEFINITIONS AND THE LEAST TIER NUMBER GRAPH

A. Definition 1

An induced sub-graph by v_i and its neighbors in G is defined as the vertex adjacent closed sub-graph corresponding to v_i and is denoted by Q_i (=G[V(N_i)]), with calling v_i as the hub[5].

Lemma Q_i of any vertex v_i of a planar graph is 4-colorable.

Proof. From definition of Q_i we know that Q_i-v_i is an outer planar graph, which is 3-colorable

by the lemma, it is viable to color v_i in the fourth color.∎

Definition 2

A subset r_1 of V(G) is said to be a vertex independent set of G if no two vertices of r_1 are adjacent in G.

V(G) of G can be partitioned into R independent sets, namely $V(G)=r_1\cup r_2\cup \ldots \cup r_R$, $r_i\cup r_j$ is not an independent set, and $r_i\cap r_j=\emptyset, 0<i<j\leq R$.

Definition 3

The independent sets are classified into two types, good and bad. The independent set r_1 is good, if and only if $\chi(G-r_1) =\chi(G)-1$; and the all independent sets of G are good if and only if the number R of independent sets is minimum, i.e. $R=\chi(G)$.

If in G there is a vertex v_i of $d(v_i)= |V(G)|-1$,then $G=Q_i$, $\chi(Q_i)=\chi(G)$, and r_1 has only the vertex v_i, so $\chi(G-v_i)=\chi(G)-1$, and the r_1 is obviously good.

Different independent set partitioning method divides the number R of independent sets of G to be different.

Take a wheel figure W_{10} with vertex label 0,1,…,10 as example, the hub vertex v_0 of $d(v_0)=\Delta(G)=|V(W_{10})|-1=10$, the rest vertices of each degree to be $\delta(G)=3$. If v_0 is divided into r_1, $r_1=\{v_0\}$ and $Qr_1=G$; then $\chi(G-r_1)=\chi(G)-1$, the r_1 is obviously good.

W_{10}-v_0 is a bipartite. we get easy that $r_2=\{v_1,v_3,v_5,v_7,v_9\}$ and $r_3=\{v_2,v_4,v_6,v_8,v_{10}\}$.

The independent set number R of W_{10} =3. i.e. $\chi(W_{10})=3$. thus, the got r_1 is obviously good by this partitioning way.

If non-adjacent vertices of degree=$\delta(W_{10})=3$ are, step by step, divided into r_1, got the $r_1=\{v_1,v_3,v_6,v_9\}$, $Qr_1=G$; Then, from W_{10}-r_1, non-adjacent vertices v_2,v_4, v_8 ,v_{10} are divided into r_2, got the $r_2=\{v_2,v_4,v_8,v_{10}\}$;Then from W_{10}-r_1-r_2, we get $r_3=\{v_5,v_7\}$, $r_4=\{v_0\}$; that are the all independent sets of W_{10}, i.e. $V(W_{10})=r_1 \cup r_2 \cup r_3 \cup r_4$. By this partitioning the independent set number R of W_{10} = 4. thus, the got r_1 is obviously bad.

Thus it can be seen that in order to get the first independent set r_1 to be good, then any $v_i\in G$ of $d(v_i)=\Delta(G)$ must first be divided into r_1, and then non-adjacent vertices with degrees from the large to the small must be, step by step, divided into r_1, until $Qr_1=G$. The r_1 is good obtained by this partition method.

If you want that V (G) of a planar graph G to be divided into 4 independent sets, then the first independent set r_1 must be good, that is, $\chi(G-r_1)=3$. Therefore, it is particularly important to get the first independent set r_1 from a planar graph, if it is bad, the future work is done in vain! Therefore, the cause of the first independent set r_1 being bad should be found first and to be resolved so as to ensure that the r_1 is good.

B. The minimum tier number graph T_k

- to construct the minimum tier number graph T_k of the given G

  The vertex degree of each vertex in the given G is firstly calculated. Then, with any v_i of $d(v_i)=\Delta(G)$ as the starting point, the tier graph with the minimum number of tiers to the starting point v_i is constructed, which is denoted by T_k. There is only one vertex v_i on T_0 tier of T_k, all vertices adjacent to v_i constitute T_1 tier, and all vertices adjacent to the vertices on T_1 constitute T_2 tier,…, all vertices adjacent to the vertices on T_{e-1} constitute the ended tier T_e.

- Characteristic of T_k is that the vertex on T_0 is adjacent only to vertices on T_1. And the vertices on T_k , 0<k<e, are adjacent only to those on T_{k-1},T_k and T_{k+1}, not adjacent to those on other tiers; and no vertex on T_{k-1} is adjacent to ones on T_{k+1}.

IV. THE NECESSARY AND SUFFICIENT CONDITIONS OBTAINED THE FIRST SET R_1 FROM A GRAPH

A $v_j \in T_2$ whose $d(v_j)=d(v_{j\text{一}})+ d(v_{j\text{二}})+ d(v_{j\text{三}})$, which is respectively the vertex number that v_j is connected ones on T_1, T_2 and T_3; $Q_{j\text{一}}$,$Q_{j\text{二}}$ and $Q_{j\text{三}}$ is respectively the induced sub-graph by $d(v_{j\text{一}})$ vertices on T_1 ,by $d(v_{j\text{二}})+1$ (v_j) vertices on T_2 and by $d(v_{j\text{三}})$vertices on T_3; there must exist $Q_{j\text{一}}\cap Q_{j\text{二}}$; there no $Q_{j\text{一}}\cap Q_{j\text{三}}$. $\chi(Q_{j\text{一}})$ and $\chi(Q_{j\text{三}})$ all≤$\chi(G)$ -1; $\chi(Q_{j\text{二}}) \le \chi(G)$.

A. The things must be done before proving following theorem 2:

- to construct T_k of the given G;
- $d(v_{j\text{一}})$ and $d(v_{j\text{二}})$ of each vertex $v_j \in T_2$ (and its following up paragraphs $T_{2,3}, T_{2,4}, \ldots, T_{2,e-1}, T_{2,e}$) must be computed out.

B. theorem 2

The first independent set r_1 of G is good, if the following conditions must be satisfied:

C1: the first vertex in r_1 must be the starting vertex v_i of T_k, whose $d(v_i)=\Delta(G)$.

C2: the next vertices partitioned into r_1 must only first be vertices$\in T_2$, and the following next ones are such vertex that gradually move to $T_{2,k}$ from T_k,2<k≤e.

C3: a vertex $v_j \in T_2$ is partitioned into r_1 such that r_1 is good, if and only if:

$d(v_{j\text{二}})=0$ of a $v_j \in T_2$. we partition the v_j whose $d(v_{j\text{二}})=0$ into r_1, then move its $d(v_{j\text{三}})$ neighbors$\in T_3$ to $T_{1,3}$, $G=Qr_1$, and the got $r_1=\{v_i,v_j\}$ is good. As $Q_{j\text{一}} \subseteq T_1$, $Q_{j\text{二}}=v_j$, $G-r_1=Qr_1-v_i-v_j= T_1 \cup Q_{j\text{三}}$, and $\chi(T_1)$ and $\chi(Q_{j\text{三}})$ all $\le\chi(G)$-1, so $\chi(G-r_1)\le\chi(G)$-1.

$d(v_{j\text{二}})>0$ of a $v_j \in T_2$. then taking the v_j and each of its $d(v_{j\text{二}})$ neighbors$\in T_2$ as hub, respectively; compute each $\chi(Q_{h\text{一}})$, so long as $\chi(Q_{h\text{一}})=3$(i.e. $Q_{h\text{一}}$ is an odd cycle) appears, right away, partition the v_h into r_1; no matter $\chi(Q_{h\text{二}})$ and $\chi(Q_{h\text{三}})$ are maximum or not, $G=Qr_1$, the got $r_1=\{v_i,v_h\}$ is good.

If all $\chi(Q_{h\text{一}})<3$,then gradually partition $v_h \in T_2$ whose $\chi(Q_{h\text{一}})$ from 2 to 1 into r_1, until $Qr_1=G$. The got r_1 must be good.

We show that if one of 3 conditions does not meet, got the first independent set r_1 may be bad; but 3 conditions all meet, got the r_1 must be good by examples.

if C1 does not meet, i.e. first regard v_i of $d(v_i)<\Delta(G)$ as the start and make up T_k, and partition it to r_1 , got the first independent set r_1 may be bad. See the example above of W_{10}.

Again, if the C1 does not satisfy, especially, if a vertex v_i whose $d(v_i) \equiv 1 \pmod 3$ is an odd number (such as 7,13,19…) and$<\Delta(G)$, is regarded as starting vertex and make up T_k, and first is partitioned the v_i into r_1(see Fig.1). When each of vertices$\in T_2$ is regarded respectively as the hub, and each $\chi(Q_{h\text{一}})$ equal to =2; and there is a vertex $v_s \in T_2$ of degree $\Delta(G)$, without loss of generality, suppose that $d(v_s)= d(v_i)+1$. And the v_s is adjacent to all vertices on T_2, see Fig.1. If again partition v_s into r_1, $G=Qr_1$,then got the $r_1=\{v_i,v_s\}$is bad(see Fig.1-1)!

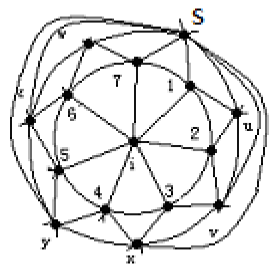

Fig.1 G $d(v_i)\neq\Delta(G)$

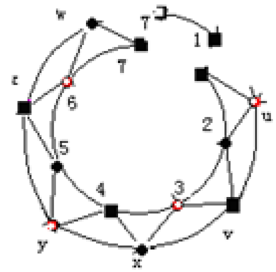

Fig.1-1 $r_1=\{v_i,v_s\}$, $H=G-r_1$,$\chi(H)=4$

Because in this case the $Q_i\cap Q_s$ is an edge in $H=G-r_1$,if the edge is decomposed from H which is decomposed into an edge and an outer planar graph ω (see Fig.1-1), although they are all 3-colorable; but with coloring the ω, the two end-vertices of the edge must be colored in the same color (see Fig.1-1, v_1 and v_7), so one of them must be colored with the fourth color that $\chi(H)=4$. So the r_1 is bad.

But, in this case if C1,C2 and C3 are all satisfied, i.e. regard a vertex of degree $\Delta(G)$ as starting vertex make up T_k, and first partition it into r_1,then partition such vertices$\in T_2$ into r_1 that each hub's $\chi(Q_{h\rightarrow})$ is the maximum, until G=Qr_1,the r_1 must be good.

Such as, in Fig.1 regard v_s of degree $\triangle(G)$ as the start and make up T_k, and first partition it into r_1,then partition $v_2,v_6\in T_2$ into r_1 (since each of them is regarded respectively as the hub, the $d(v_{2\rightarrow})$ and $d(v_{6\rightarrow})=3$; $\chi(Q_{2\rightarrow})$and $\chi(Q_{6\rightarrow})=2$,they are all the maximum in vertices $\in T_2$); then move vertices $v_i,v_3,v_5\in T_2$ into $T_{1,2}$ (since they are adjacent to v_2 or v_6), at this time, only a vertex v_4 remains on T_2,so partition $v_4\in T_2$ to r_1, G=Qr_1, got the $r_1=\{v_2,v_6,v_4,v_s\}$ is good.

Because in H=G-r_1 there are vertices of degree 2(in Fig.1-2 the vertices u and w of degree 2),it suffices to split each of them into two vertices so that the H become a outer planar graph with suspended vertices u, w, and its $\chi(H)=3$,see Fig.1-2.

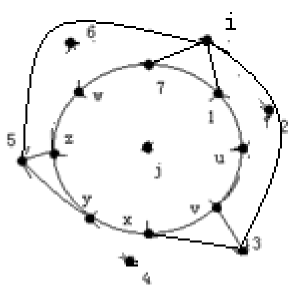

Fig.1-2 H=G-r_1,$\chi(H)=3$

In addition, if 3 conditions all satisfy, i.e. regard a v_i of degree $\triangle(G)\equiv1(\text{mod}3)$ being odd as the start and make up T_k, and first partition it into r_1 ,even if there is another vertex $v_j\in T_2$ of degree $\triangle(G)\equiv1(\text{mod }3)$ to be odd ,see Fig.2 the vertices v_i and v_j of degree $\triangle(G)$,again partition v_j and its non-adjacent vertices $\in T_2$ into r_1, G=Qr_1, got the $r_1=(v_i,v_j)$ is also good.

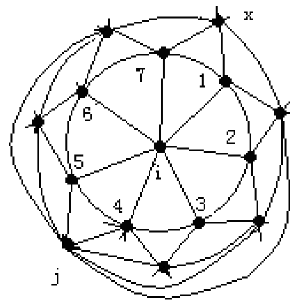

Fig. 2 G

Because v_j is adjacent to vertices $\in T_1$ of degree 1, it means that v_j is adjacent to two vertices $\in T_1$,so it is adjacent to at most $\triangle(G)$-2 vertices $\in T_2$, that is, in T_2 there exist at lest two vertices which can partition into r_1, then partition those vertices into r_1, G=Qr_1, got the r_1 is good, see Fig.2 and 2-1. got the $r_1=\{v_i,v_j,x\}$is good.

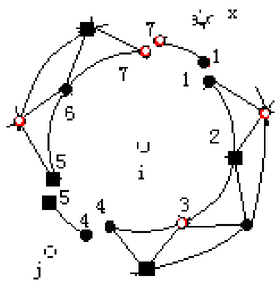

Fig.2-1

Since the $Qr_1\cap Q_j$ is two edges in H(in Fig.2-1 $e_{1,7}$ and $e_{4,5}$).Split such two edges from H that H become two outer planar graphs and two edges. And 4 vertices connected by the two edges belong respectively to two distinct outer planar graphs(in Fig.2-1 vertices v_1 and v_4 belong to an outer planar, vertices v_5 and v_7 belong to the other),even if the two vertices of degree 2 in an outer planar graph must be colored with the same color(say, in Fig.2-1 the vertices v_1 and v_4), the H= G-r_1 is also 3-colorable.It is viable that with coloring two outer planar graphs, need merely to color two pairs vertices in distinct outer planar graphs with different colors, i.e. the endpoints of the two edges with different colors .

Why do we have C2?

Two reasons: first, from T_2 layer its subsequent segments $T_{2,x}$, $2<x\leq e$, select vertices who satisfy C2, and to divide

them into r_1, until $G=Qr_1$; so that vertices that are not adjacent to the starting point are not missed;

The other is that vertices of $k\geq3$ layers are divided into r_1, may be in large probability , such that $G\neq Qr_1$!and got the $r_1$must be bad.

The reasons that C3 needs be satisfied are shown follows below:.

V. FOR EXAMPLE

FINALLY, WE SHOW IN DETAIL THE ENTIRE PROCEDURE TO PROVE THE 4CT BY PARTITIONING 25 VERTICES OF A PLANAR G IN FIG.A INTO FOUR INDEPENDENT SETS. BY WHICH IN 1890 P. J. HEAWOOD OVERTHROWN THE PROOF OF THE 4CT BY A. B. KEMPE IN 1879.

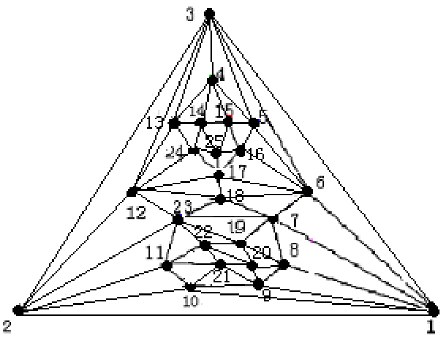

Fig. A. G on 25 vertices

show: *1.1* First compute each vertex degree in Fig. A. G. Vertices of degree $\Delta(G)=7$ are: $v_1,v_3,v_6,v_{12},v_{23}$; vertices of degree 6 are: v_2,v_7,v_{17}, the remains vertices of degree 5.

1.2. Select arbitrarily a vertex v_i whose $d(v_i)=\Delta(G)=7$ of G in Fig. A, like v_3, regard v_3 as the starting make up tier graph T_k (see Fig.A-1), and partition it into r_1.

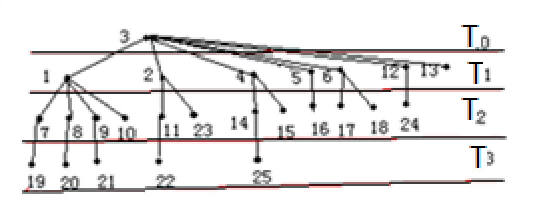

Fig.A-1 the T_k taking v_3 as the start

1.3. For manual analysis convenience, T_k in Figure A-1 is shown in Figure A-2.

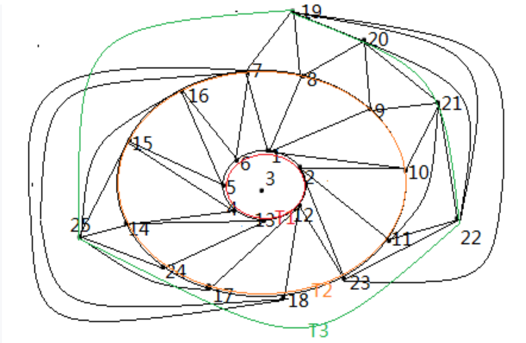

Fig.A-2 the T_k

By observing, there is no vertex of $d(v_j)=0$.vertices of $d(v_j)=2$, its $\chi(Q_j)=2$, are v_7,v_{10},v_{23}, v_{14},v_{15},v_{16} . So, first, partition v_7 into r_1,move its adjacent vertices $v_8,v_{18},v_{23}\in T_2$ to $T_{1,2}$; move $v_{19}\in T_3$ to $T_{1,3}$; then partition v_{10},v_{14},v_{16} into r_1.move its adjacent vertices $v_9,v_{11};v_{15},v_{24};v_{17}\in T_2$ to $T_{1,2}$; move $v_{21},v_{25}\in T_3$ to $T_{1,3}$; move $v_{20},v_{22}\in T_3$(they adjacent to the vertices on $T_{1,2}$) to $T_{2,3}$.

*1.4.*on $T_{2,3}$ vertices v_{20} and v_{22} of $d(v_j)=2$, its $\chi(Q_j)=2$; so partition one of both, such as v_{22} into r_1,at here $G=Qr_1$, got $r_1=\{v_3,v_7,v_{10},v_{14},v_{16},v_{22}\}$ and $H=G-r_1$.see Fig. B.

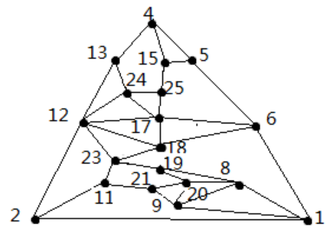

Fig. B. $H=G-r_1$

2.1 First compute each vertex degree of G in Fig.B. v_{12} of degree $\Delta(H)=6$, v_{17} of degree 5,the vertices of degree 4 are $v_1,v_6,v_8,v_9,v_{18},v_{20},v_{23},v_{24}$, the remaining vertices $v_4,v_5,v_{15},v_{13},v_{25},v_{11},v_{19},v_{21},v_2$ of degree 3.

2.2 Regard v_{12} of degree $\Delta(H)=6$ as the starting vertex make up T_k (see Fig.B-1.). And partition it to r_2.

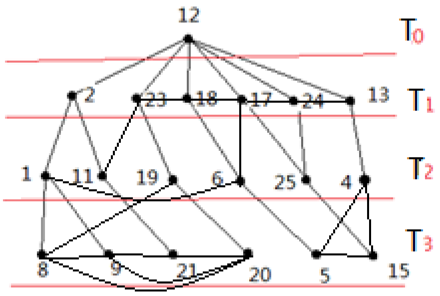

Fig.B-1. T_k taking v_{12} as the staring

2.3 vertices of $d(v_j)=0$ are v_{11},v_{19},v_{25},v_4, so simply partition them into r_2,then move their adjacent vertices v_{20},v_{21},v_8 and $v_5,v_{15}\in T_3$ to $T_{1,3}$.Since v_1 of $d(v_1)=1$, its $\chi(Q_1)=1$, but v_6 of $d(v_6)=2$, its $\chi(Q_6)=2$. so partition the v_6 into r_2,then move its adjacent vertices v_1 to $T_{1,2}$, move $v_9\in T_3$ to $T_{2,3}$.

2.4 since there is only vertex v_9 on $T_{2,3}$, so, partition it into r_2, at here $H=Qr_2$, so we got that $r_2=\{v_{12},v_6,v_4,v_{11},v_{25},v_{19},v_9\}$ and $H'=H-r_2$.

H' is a bipartite with an $e=(v_5,v_{15})$ and 2 paths: $P_1=\{v_2,v_1,v_8,v_{20},v_{21}\}$, $P_2=\{v_{13},v_{24},v_{17},v_{18},v_{23}\}$.

3. it is easy that partition V(H') into 2 independent sets: $r_3=\{v_{23},v_{17},v_{13},v_1,v_{20},v_5\}$ and

$r_4=\{v_{18},v_{24},v_2, v_8, v_{21},v_{15}\}$.

From this example one can see that using the partitioning independent sets way can partition V(G) of a planar graph into 4 independent sets by one time operation. It not only the 4CT, but also can get the colors at the every vertex of the graph.

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