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Abstract

An R module M is said to be epi-Artinian if for every descending chain $M_1 \geq M_2 \geq \ldots$ of submodules of M, there exists an index n such that M_{i+1} is homomorphic image of $M_i, \forall i \geq n$. In this paper, we discuss some properties of epi-Artinian rings and modules. We characterize epi-Artinian modules with iso-Artinian modules. We also discuss some properties of iso-Artinian rings and modules.

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1 Introduction and Preliminaries

In this paper, all rings are associative with unit element and all modules are unitary right modules.

In [7], Facchini et. al. defined the notion of iso-Noetherian and iso-Artinian modules. According to them, a module M iso-Noetherian (iso-Artinian) if for every ascending (descending) chain $M_1 \leq M_2 \leq M_3 \leq \ldots (M_1 \geq M_2 \geq M_3 \geq \ldots)$ of submodules of M, there exists an index n such that M_n is isomorphic to M_i for every $i \geq n$. A ring R is said to be right iso-Noetherian (right iso-Artinian) if the right module R_R is iso-Noetherian (iso-Artinian).

In [12], we define epi-Artinian module. We say that a modules M is epi-Artinian if for every descending chain $M_1 \geq M_2 \geq M_3 \geq \ldots$ of submodules of M, there exists an index n such that M_{i+1} is a homomorphic image of M_i for every $i \geq n$. In [5], the authors call epi-dcc this chain condition. We say that a ring R is right epi-Artinian if the right module R_R is epi-Artinian.

Every iso-Artinian module is epi-Artinian, but epi-Artinian modules need not be iso-Artinian. We give examples of an epi-Artinian module which is not iso-Artinian. We provide sufficient conditions for epi-Artinian modules to be iso-Artinian. If R is an integral domain, then R_R is iso-Artinian if and only if R_R is epi-Artinian.

2 Epi-Artinian and Iso-Artinian Rings and Modules

We begin this section with the following lemma.

Lemma 2.1. *Let R be an iso-Artinian ring. Then R contains a uniform ideal.*

Proof. If R is uniform, then we are done. If not, R contains a direct sum of nonzero ideals, say $R = I_0 \supseteq I_1 \oplus I_1'$. If either of I_1, I_1' is uniform, then we are done. If not, repeating this argument for I_1, we get I_2, I_2' such that $I_1 \supseteq I_2 \oplus I_2'$. If either of I_2, I_2' is uniform, then we are done. If not, repeating the process for $I_3, I_4, \ldots$, we get a direct sum $I_1' \oplus I_2' \oplus I_3' \oplus \ldots$. The finite uniform dimension shows that this process must stop at a finite step k. At this stage the ideal I_k is uniform. □

In the following lemma, we we discuss structure of essential right ideal of a right iso-Artinian ring in terms of uniform right ideals.

Proposition 2.2. *If R is a right iso-Artinian ring. Then R contains an essential right ideal which is a finite direct sum of uniform right ideal.*

Proof. Let $I' = \oplus_{i=1}^{n} U_i$ be a direct sum of uniform right ideals U_i of R. Suppose that I' is not essential in R. Then there exists a nonzero right ideal J of R such that $I' \cap J = 0$. By lemma 2.1, J contains a uniform right ideal, say U_{n+1} and $R \supseteq I' \oplus U_{n+1} = \oplus_{i=1}^{n+1} U_i$. If $\oplus_{i=1}^{n+1} U_i$ is not essential in R, then repeating this process, either we get an essential submodule or else an infinite direct sum, which is not possible because R has finite uniform dimension. □

Remark 2.3. *[12, Remark 3.8] Every iso-Artinian module is epi-Artinian. But, in general, epi-Artinian modules need not be iso-Artinian. For example, let $M = \oplus_{p \in \mathbb{P}} \mathbb{Z}_p$, where $\mathbb{P}$ be the set of all prime integers. Then M is epi-Artinian, but not iso-Artinian. In the following, we provide sufficient conditions for epi-Artinian modules to be iso-Artinian.*

Proposition 2.4. *Let M be a uniform torsion free R-module. Then M is epi-Artinian if and only if M is iso-Artinian.*

Proof. Let $M_1 \geq M_2 \geq M_3 \geq \ldots$ be a descending chain of submodules of M. Since M is epi-Artinian, there exists an index n such that M_{i+1} is homomorphic image of M_i, for all $i \geq n$. Let $f : M_i \to M_{i+1}$ be epimorphism then f is an endomorphism of M_n. Since M_n is uniform and torsion free hence satisfies (*)-property. Thus f is monomorphism. Therefore f is isomorphism. □

Proposition 2.5. *Let R be a right epi-artinian ring. If every nonzero right ideal of R contains a right regular element, then R is right Noetherian.*

Proof. It is sufficient to show that every rihgt ideal of R is finitely generated. On contrary, let I be a non finitely generated right ideal of R. Let $x \in I$ be a right regular element. Then $R \cong xR$ and xR contains a right ideal xI, which is isomorphic to I as a right R- module. Thus we construct a descending chain $R \geq I \geq xR \geq xI \geq x^2R \geq x^2I \geq \ldots$, where $x^nI \cong I$ is not finitely generated and $x^nR \cong R$ is finitely generated. Since R is epi-artinian, there exists an index m such that x^iI is epimorphic image of x^iR, for all $i \geq m$. This shows that $x^iI \cong I$ is finitely generated, a contradiction. Thus I is finitely generated. Therefore R is right Noetherian. □

Recall by [2] that an R-module M is vertually semisimple if every submodule of M is isomorphic to a direct summand of M. If every submodule of M is vertually semisimple then M is said to be completely vertually semisimple.

Lemma 2.6. *Let R be a semiprime iso-Artinian ring. Then every projective R-module M is completely virtually semisimple.*

Proof. Since R is semiprime iso-Artinian ring. Therefore R is direct sum of iso-retractable ideals, by [12, Proposition 2.7]. Now by [2, Theorem 3.11], R is a left completely virtually semisimple ring. Thus by [2, Proposition 3.3], every projective left R-module is completely virtually semisimple. □

We know that every right Artinian ring is right Noetherian. In the following, we show that under semiprimeness condition iso-Artinian ring becomes right Noetherian.

Corollary 2.7. *Let R be a semiprime iso-Artinian ring. Then R is a right Noetherian ring.*

Proof. By Lemma 2.6, R is completely virtually semisimple and $u.dim(R) < \infty$. Thus [2, Proposition 2.8] implies that R is right Noetherian ring. □

Proposition 2.8. *Let R be a semiprime iso-Artinian hereditary ring. Then every finitely generated projective R-module is iso-Artinian.*

Proof. By Lemma 2.6. □

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