Cyclocopula Technique to Study the Relationship Between Two Cyclostationary Time Series with Fractional Brownian Motion Errors

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Abstract. Detection of relationship between two time series is so important in environmental and hydrological studies. Several parametric and non-parametric approaches can be applied to detect relationships. These techniques are usually sensitive to stationarity assumption. In this research, a new copula- based method is introduced to detect the relationship between two cylostationary time series with fractional Brownian motion (fBm) errors. The numerical studies verify the performance of the introduced approach.

Keywords: Time Series; Fractional Brownian Motion, Cyclostationary, Copula, Regression.

1. Introduction

The ways of modeling dependency between two time series has always been one of the main focuses in practice. The choice of the better model depends on the dependency structure of obligators and is crucial part of the modeling. The applied methods in previous studies include Pearson's correlation coefficient [1-6], Spearman's correlation coefficient [7-10], Kendall's correlation coefficient [11-14], Sen's slope [15-17], cross-correlation function [18-20] and copula [21-26]. When we face with the relationship of two stationary time series, cross-correlation function and copula are suggested. Cross-correlation function, as same as Pearson's correlation, is somewhat sensitive to abnormality of datasets and existence of outliers. In other words, for abnormal populations or when we face with outliers, cross-correlation function may not work well. An efficient way to model dependency is to use new modeling mechanism of Copula Theory which helps understand the correlation beyond linearity. In contrast, Copula is a function which transfers the multivariate distribution function to its marginal distribution function by quantiles. It is a great tool for modeling dependency where correlation follows the random distribution. Copula technique is most efficient for stationary time series and may not

work well for non-stationary time series such as cyclostationary time series. To solve this issue, in this research, we introduce a copula-based regression technique. The ability of the proposed approach to detect relationship between two cyclostationary time series with fBm errors is studied. For this purpose, numerous datasets from two cyclostationary time series with fBm errors are produced and analyzed.

2. Methodology

2.1. Copula

Copula was first introduced by Sklar [27] as a statistical mechanism to transfer the joint distribution into its marginals and copula as a model to show the dependency between the marginals.

Copulas are functions link marginal distributions to the multivariate distributions which have well-defined properties [28]. Assume X and Y are two continuous random variables with a joint distribution function

$$H(x,y) = Pr(X \leq x, Y \leq y),$$

and marginals

$$F(x) = Pr(X \leq x),$$

and

$$G(y) = Pr(Y \leq y).$$

According to Sklar's theorem [27], there is a copula for all x and y in $[-\infty, \infty]$ satisfies the following equation:

$$H(x,y) = C\big(F(x), G(y)\big).$$

The theorem indicates that copula is joint distribution function, and joint distribution function can be also presented as copula given its marginal distributions. Thus, Schweizer [29] discussed that joint distribution modeling can be reduced to copula modeling. Since copula represents the variables' dependence, it's also named dependence function.

One of the main advantages of copulas is allowing to identify tail dependence across the multiple distributions. There are several Copulas that can be selected. The choice of most appropriate Copula has been an important issue. In practice, independent and perfectly correlated variables are naturally inapplicable for copulas. Due to its familiarity, Gaussian Copula [30] was the most famous one among others, however it failed to capture asymmetry, non-linearity and heavy tail dependency.

Therefore, alternative copulas have been used to model joint dependences. Several papers have been implied Copula families in modeling joint default dependencies in hydrology. Gumbel copula [31] is able to capture right tail dependence which is a particularly explored aspect of default dependency. Clayton copula [32] is defined to have left tail dependence. Student's t copula [33] captures symmetric tail dependence with equally right and left tail dependence while Gaussian and Frank copulas [34] are defined as symmetric dependence without any tail dependence.

2.2. Cyclostationary Time Series

Stationarity is an important condition in classical time series analysis. But because of existence of cyclic rhythm in many practical situations such as hydrology and climatology, the stationarity assumption is not satisfied. As a suitable alternative, cyclostationary (CS) time series [35-36] are employed to describe the cyclic rhythms of rhythmic processes.

A time series X_t, is CS with cycle T (CS-T), if T is the minimum natural number so that

$$m(t) := E(X_t) = m(t + T),$$

and

$$R(s,t) := Cov(X_s, X_t) = E\left[(X_s - m(s))\overline{(X_t - m(t))}\right] = Cov(X_{s+T}, X_{t+T}),$$

for all integers s and t.

The cyclic correlation can be detected by employing the coherent statistics [36]. These statistics were defined by

$$|\hat{\gamma}(p,q,M)|^2 = \frac{\left|\sum_{m=0}^{M-1} d_X(\lambda_{p+m})\overline{d_X(\lambda_{q+m})}\right|^2}{\sum_{m=0}^{M-1}\left|d_X(\lambda_{p+m})\right|^2 \sum_{m=0}^{M-1}\left|d_X(\lambda_{q+m})\right|^2}, \quad p > 0, q \leq N,$$

where

$$d_X(\lambda) = n^{-1/2} \sum_{t=1}^{n} X_t e^{i(t-1)\lambda} , \lambda \in [0,2\pi),$$

denotes the discrete Fourier transform (DFT) of $X_1, \dots, X_n$, M refers to the the smoothness parameter, and

$$(\lambda_i, \lambda_j) \in \bigcup_{k \in \mathbb{Z}} \left\{ (\lambda_i, \lambda_j) \in [0,2\pi)^2 : \lambda_j = \lambda_i + \frac{2\pi k}{T} \right\}.$$

Since the spectral domain of CS-T time series is supported on the lines

$$T_j(x) = x \pm \frac{2\pi(j-1)}{T}, j = 1, \dots, T,$$

therefore, we expect the coherent statistics plot of a CS process emphasis these lines.

Because of cyclic rhythms in CS processes, using usual copula regression to compute the relationship of two CS-T time series is somewhat wrong. To solve this issue, in this research, a new approach is employed to evaluate relationship of two CS-T time series.

2.3. Fractional Brownian Motion

A fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$, is defined by

$$B_H(t) = \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \int_0^t (t-s)^{H-\frac{1}{2}} dB(s),$$

where B and Γ are a Brownian motion process and gamma function [37-38], respectively. The auto-covariance function of the processes $B_H(t)$ and $B_H(s)$ are given by

$$\gamma(s,t) := Cov\left(B_H(s), B_H(t)\right) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

2.4. Copula for Cyclostationary Time Series

As previously discussed, the copula technique is most efficient for stationary time series and may not work well for non-stationary time series such as cyclostationary time series. To solve this issue, in this research, we introduce a copula-based regression technique.

Assume X_t and Y_t are two CS-T time series. Let $\{x_1, \ldots, x_n\}$ and $\{y, \ldots, y_n\}$ $(n = mT, m \in N)$ are a path of X_t and Y_t, respectively. The outline as our procedure as following:

(i) Split $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ and $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ into T partitions $\{x^{(1)}, \ldots, x^{(T)}\}$, $\{y^{(1)}, \ldots, y^{(T)},\}$ and $\{(x, y)^{(1)}, \ldots, (x, y)^{(T)}\}$, where

$$x^{(i)} = \{x_i, x_{i+T}, \ldots, x_{i+(m-1)T}\}, i = 1, \ldots, T,$$
$$y^{(i)} = \{y_i, y_{i+T}, \ldots, y_{i+(m-1)T}\}, i = 1, \ldots, T,$$

and

$$(x, y)^{(i)} = \{(x_i, y_i), (x_{i+T}, y_{i+T}), \ldots, (x_{i+(m-1)T}, y_{i+(m-1)T})\}, i = 1, \ldots, T.$$

(ii) Let F_i, G_i and H_i as the distribution function of the members of $x^{(i)}$, $y^{(i)}$, and $(x, y)^{(i)}$, respectively. Define the copula of $x^{(i)}$ and $y^{(i)}$ by

$$C_i = C(F_i, G_i), i = 1, \ldots, T,$$

and estimate the copula of $x^{(i)}$ and $y^{(i)}$ by

$$\hat{C}_i = \hat{C}(F_i, G_i), i = 1, \ldots, T.$$

(iii) Apply copula regression to find the regression equation of $y^{(i)}$ based on $x^{(i)}$,

$$\hat{y}_j = b_{0,i} + b_{0,i} x_j, i = 1, \ldots, T, j = i, i + T, i + (m - 1)T.$$

(iv) Combine the regression equations to next uses such as prediction or goodness of fit tests.

Remark 1: If X_t and Y_t are respectively $CS - T_1$ and $CS - T_2$ time series, let $T = lcm(T_1, T_2)$, where lcm refers to smallest common multiple of T_1 and T_2.

3. Simulation

In this section, the ability of the proposed method to detect relationship between two CS time series is studied. For this purpose, numerous datasets from two CS time series X_t and Y_t are produced and analyzed.

The simulation procedure is as following:

Step 1: For fixed $n \in \{120, 240, 480, 1200\}$, $H \in \{0.25, 0.75\}$ and $T \in \{1, 2, 3, 4\}$, separate paths of size n from two CS-T time series X_t and Y_t are produced. Numerous CS time series with different parameters are considered.

Step 2: The simulated dataset are split into T partitions $\{x^{(1)}, ..., x^{(T)}\}$ and $\{y^{(1)}, ..., y^{(T)}\}$. Then the copula of $x^{(i)}$ and $y^{(i)}$ is estimated by

$$\hat{C}_i = \hat{C}(F_i, G_i), i = 1, ..., T.$$

In this study, we apply five different copula families, namely Gaussian, Student's t copula and three Archimedean copulas; Clayton, Gumbel and Frank. We capture symmetric dependence without tail dependence with Gaussian and Frank copulas, symmetric dependence with upper and lower tail dependence with t copula, left (lower) tail dependence with Clayton copula, right (upper) tail dependence with Gumbel copula.

3.1. Gaussian Copula

Gaussian (normal) copula, as the name implies, assumes joint distribution follows bivariate standard normal distribution. Gaussian copula is the most commonly applied copula in practice due to its convenient properties. The bivariate Gaussian copula is presented by:

$$C(a, b) = M_n(\Phi^{-1}(a), \Phi^{-1}(b); \theta),$$

where M_n is the joint bivariate cumulative standard normal distribution with $\theta \in [-1, 1]$, as the correlation of the bivariate normal distribution and Φ^{-1} is the inverse of a univariate standard normal distribution.

3.2. t Copula

The bivariate Student's t copula is represented by:

$$C(a, b) = T_{\theta,v}\big(t_v^{-1}(a), t_v^{-1}(b)\big),$$

where $T_{R,v}$ is the standardized bivariate Student's t distribution with covariance $\theta \in [-1,1]$ and v degree of freedom. $t_v^{-1}(u_n)$ indicates the inverse of Student's t cumulative distribution function. The main advantage of Student's t copula over Gaussian copula is assuming a non-zero tail dependence even if correlation is zero.

3.3. Clayton copula

To estimate the lower tail dependency, the Clayton copula, is mostly suggested. The bivariate Clayton copula is represented by:

$$C(a, b) = \Big(max\big(a^{-\theta} + b^{-\theta} - 1, 0\big)\Big)^{-1/\theta},$$

where θ ($\theta \geq -1, \theta \neq 0$) is the copula. The parameter θ is related to Kendall's tau rank correlation τ as following:

$$\tau = \frac{\theta}{\theta+2}.$$

3.4. Gumbel copula

To capture weak lower tail dependence and strong upper tail dependence, the Gumbel copula is developed. The bivariate Gumbel copula is given by:

$$C(a, b) = \exp\Big(-\big((-\log a)^\theta + (-\log b)^\theta\big)^{\frac{1}{\theta}}\Big),$$

where $\theta \geq 1$ is the copula parameter. The Gumbel copula represents just independent and positive dependence.

The parameter θ is related to Kendall's tau rank correlation τ as following:

$$\tau = 1 - \theta^{-1}.$$

3.5. Frank copula

The bivariate Frank copula is given by:

$$C(a, b) = -\frac{1}{\theta} \log \left(1 + \frac{(exp(-\theta a)-1)(exp(-\theta b)-1)}{exp(-\theta)-1} \right),$$

where $\theta \neq 0$ is the copula parameter. Unlike the Gumbel and Clayton copulas, Frank copula allows both negative and positive dependence in data. The parameter θ is related to Kendall's tau rank correlation τ as following:

$$\tau = 1 + \frac{4[D_1(\theta)-1]}{\theta},$$

where

$$D_k(\alpha) = \frac{k}{\alpha^k} \int_0^\alpha \frac{t^k}{\exp(t) - 1} dt, k = 1,2.$$

Step 3: For each copula, the regression analysis is applied to estimate the equation of $y^{(i)}$ based on $x^{(i)}$,

$$\hat{y}_j = b_{0,i} + b_{0,i} x_j, i = 1, \dots, T, j = i, i + T, i + (m-1)T.$$

Step 4: For each copula, the estimated copula regression equations are used to estimate T partitions $\{y^{(1)}, \dots, y^{(T)}\}$ by $\{\hat{y}^{(1)}, \dots, \hat{y}^{(T)}\}$.

Step 5: Different goodness of fit measures including Correlation coefficient (r), Willmott's Index (WI) and Nash-Sutcliffe coefficient (NS) are computed by

$$r = \frac{\sum_{i=1}^n (y_i - \bar{y})(\hat{y}_i - \bar{\hat{y}})}{\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2} \sqrt{\sum_{i=1}^n (\hat{y}_i - \bar{\hat{y}})^2}},$$

$$WI = \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{\sum_{i=1}^n (|y_i - \bar{y}| + |\hat{y}_i - \bar{y}|)^2},$$

and

$$NS = 1 - \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{\sum_{i=1}^n (\hat{y}_i - \bar{\hat{y}})^2}.$$

Step 6: Steps 1 to 5 are repeated 1000 times.

Step 7: For each parameter setting, the means of r, WI and NS of all 1000 runs and five copulas are computed.

We consider first order periodic autoregressive with fBm error (PARFBM(1)) time series.

Assume the process

$$X_t = \phi(t)X_{t-1} + B_H,$$

and

$$Y_t = \alpha X_t + W_t, \qquad \{W_t\} \sim IIDN(0,1),$$

where

$$\phi(t) = \frac{1 + \phi \cos(2\pi t/T)}{2}.$$

Tables 1 and 2 summarize the computed values of goodness of fit indices, for different parameter settings and $H = 0.25$ and $H = 0.75$, respectively. The results show that the values of goodness of fit indices are close to one. In other words, proposed method is robust to detect relationship between two cyclostationary time series.

Table 1: Testing performance of the Periodic Copula Model to detect relationship between two PARFBM(1) time series ($H = 0.25$), in terms of the Correlation Coefficient (r), Nash-Sutcliff (NSE) and Willmott Index (WI)

Copula	T	ϕ	α	N											
				120			240			480			1200		
				r	WI	NS	r	WI	NS	r	WI	NS	r	WI	NS
Gaussian	1	0.3	0.3	0.980	0.992	0.989	0.978	0.991	0.988	0.978	0.983	0.990	0.998	0.995	0.980
		0.3	0.7	0.977	0.989	0.998	0.985	0.992	0.992	0.970	0.978	0.998	0.983	0.997	0.997
		0.7	0.3	0.994	0.985	0.999	0.981	0.990	0.973	0.996	0.977	0.996	0.989	0.994	0.990
		0.7	0.7	0.984	0.989	0.991	0.988	0.998	0.987	0.983	0.986	0.985	0.983	0.986	0.976
	2	0.3	0.3	0.992	0.987	0.973	0.994	1.000	0.976	0.987	0.983	0.994	0.977	0.993	0.987
		0.3	0.7	0.984	0.976	0.995	0.984	0.991	0.995	0.987	0.990	0.984	0.973	0.992	0.972
		0.7	0.3	0.997	0.986	0.999	0.998	0.981	0.989	0.992	0.987	0.986	0.987	0.975	0.990
		0.7	0.7	0.970	0.972	0.978	0.988	0.992	0.979	0.982	0.993	0.985	0.971	0.986	0.978
	3	0.3	0.3	0.978	0.983	0.982	0.996	0.983	0.982	0.979	0.976	0.976	0.996	0.994	0.987
		0.3	0.7	0.970	0.984	0.975	0.981	0.992	0.975	0.975	1.000	0.987	0.983	0.982	0.971
		0.7	0.3	0.981	0.992	0.997	0.976	0.980	0.989	0.980	0.994	0.989	0.977	0.984	0.981
		0.7	0.7	0.979	0.971	0.982	0.992	0.991	0.989	0.998	0.995	0.990	0.980	0.999	0.992
	4	0.3	0.3	0.992	0.986	0.972	0.985	0.975	0.984	0.986	0.970	0.980	0.997	0.971	0.990
		0.3	0.7	0.983	0.982	0.970	0.972	0.994	0.978	0.985	0.983	0.978	0.970	0.996	0.972
		0.7	0.3	0.985	0.977	0.978	0.982	0.977	0.982	0.987	0.994	0.987	0.981	0.993	0.995
		0.7	0.7	0.989	0.989	0.980	0.977	0.972	0.983	0.972	0.989	0.996	0.993	0.972	0.997
	1	0.3	0.3	0.996	0.971	0.996	0.978	0.971	0.996	0.991	0.978	0.986	0.998	0.979	0.975

Family	Idx														
T		0.3	0.7	1.000	0.978	0.998	0.996	0.979	0.991	0.999	0.984	0.989	0.973	0.981	0.997
		0.7	0.3	0.974	0.996	0.990	0.979	0.986	0.976	0.993	0.986	0.974	0.988	0.975	0.976
		0.7	0.7	0.985	0.985	0.999	0.984	0.976	0.972	0.979	0.998	0.973	0.988	0.975	0.983
	2	0.3	0.3	0.978	0.971	0.985	0.983	0.997	0.973	0.976	0.973	0.985	0.994	0.998	0.979
		0.3	0.7	0.994	0.990	0.986	0.984	0.975	0.999	0.994	0.987	0.973	0.984	0.995	0.999
		0.7	0.3	0.983	0.970	0.997	0.981	0.983	0.999	0.993	0.972	0.987	0.971	0.970	0.995
		0.7	0.7	0.984	0.991	0.978	0.972	0.988	0.992	0.983	0.973	0.991	0.998	0.994	0.980
	3	0.3	0.3	0.993	0.979	0.978	0.983	0.992	0.993	0.979	0.978	0.985	0.977	0.998	0.971
		0.3	0.7	0.979	0.993	0.996	0.989	0.993	0.995	0.989	0.999	0.996	0.971	0.980	0.984
		0.7	0.3	0.999	0.975	0.982	0.979	0.994	0.996	0.977	0.996	1.000	0.997	0.993	1.000
		0.7	0.7	0.986	0.986	0.991	0.970	0.992	0.978	0.978	0.992	0.997	0.976	0.986	0.978
	4	0.3	0.3	0.997	0.976	0.979	0.970	0.992	0.976	0.974	0.970	0.980	0.998	0.999	0.979
		0.3	0.7	0.977	0.975	0.979	0.994	0.989	0.980	1.000	0.994	0.974	0.970	0.984	0.982
		0.7	0.3	0.986	0.983	0.984	0.992	0.980	0.976	0.978	0.987	0.981	0.973	0.982	0.986
		0.7	0.7	0.993	0.971	0.992	0.998	0.996	0.999	0.992	0.984	0.998	0.996	0.976	0.982
Clayton	1	0.3	0.3	1.000	0.979	0.985	0.977	0.987	0.984	0.980	0.991	0.986	0.991	0.987	0.986
		0.3	0.7	0.981	0.999	0.999	0.996	0.996	0.998	0.992	0.993	0.981	0.984	0.986	0.989
		0.7	0.3	0.973	0.995	0.997	0.992	0.979	0.990	0.971	0.996	0.990	0.988	0.972	0.980
		0.7	0.7	0.986	0.986	0.979	0.994	0.977	0.996	0.989	0.998	0.983	0.992	0.976	0.982
	2	0.3	0.3	0.982	0.996	0.993	0.988	0.975	0.990	0.986	0.983	0.980	0.972	0.997	0.990
		0.3	0.7	0.984	0.975	0.981	0.986	0.983	0.985	0.977	0.970	0.974	0.975	0.999	0.982
		0.7	0.3	0.989	0.992	0.978	0.981	0.971	0.995	0.995	0.975	0.994	0.982	0.971	0.987
		0.7	0.7	0.993	0.988	0.976	0.982	0.991	0.987	0.996	0.986	0.987	0.997	0.973	0.970
	3	0.3	0.3	0.988	0.978	0.975	0.987	0.989	0.994	0.995	0.982	0.972	0.984	0.989	0.983
		0.3	0.7	0.973	0.998	0.980	0.980	0.980	0.988	0.982	0.981	0.983	0.993	0.975	0.991
		0.7	0.3	0.995	0.999	0.989	0.992	0.985	0.993	0.999	0.990	0.979	0.979	0.990	0.978
		0.7	0.7	1.000	0.991	0.971	0.992	0.998	0.992	0.972	0.981	0.978	0.988	0.987	0.971
	4	0.3	0.3	0.981	0.989	0.983	0.975	0.997	0.995	0.989	0.991	0.981	0.993	0.996	0.978
		0.3	0.7	0.975	0.984	0.971	0.987	0.983	0.970	0.972	1.000	0.981	0.989	0.987	1.000
		0.7	0.3	0.971	0.988	0.981	0.975	0.971	0.999	0.996	0.977	1.000	0.991	0.997	0.992
		0.7	0.7	0.971	0.996	0.980	1.000	0.972	0.977	0.987	0.973	0.975	0.993	0.985	0.998
Gumbel	1	0.3	0.3	0.997	0.990	0.983	0.979	0.985	0.982	0.987	0.983	0.997	0.979	0.993	0.983
		0.3	0.7	0.972	0.974	0.984	0.971	0.972	0.997	0.989	0.983	0.997	0.978	0.997	0.985
		0.7	0.3	0.988	0.974	0.999	0.979	0.996	0.974	0.979	0.985	0.973	0.987	0.980	0.993
		0.7	0.7	0.987	0.987	0.992	0.988	0.998	0.993	0.975	0.973	0.978	0.970	0.987	0.990
	2	0.3	0.3	0.996	0.996	0.988	0.991	0.980	0.985	0.986	0.973	0.988	0.996	0.995	0.997
		0.3	0.7	0.998	0.998	0.973	1.000	0.983	0.972	0.996	0.981	0.986	0.973	0.976	0.998
		0.7	0.3	0.971	0.979	0.974	0.982	0.997	0.992	0.974	0.995	0.990	0.974	0.977	0.986
		0.7	0.7	0.975	0.993	0.995	0.980	0.992	0.987	0.994	0.984	0.972	0.997	0.970	0.971
	3	0.3	0.3	0.977	0.979	0.977	0.995	0.996	0.993	0.978	0.975	0.997	0.987	0.982	0.999
		0.3	0.7	0.983	0.999	0.975	0.995	0.984	0.992	0.993	0.971	0.974	0.998	0.970	0.972
		0.7	0.3	0.989	0.972	0.988	0.970	0.976	0.999	0.993	0.972	0.982	0.996	0.981	0.986
		0.7	0.7	0.983	0.999	0.977	0.993	0.973	0.975	0.972	0.971	0.990	0.998	0.979	0.985
	4	0.3	0.3	0.970	0.982	0.984	0.974	0.974	0.973	0.975	0.974	0.974	0.977	0.981	0.993
		0.3	0.7	0.992	0.974	0.974	0.971	0.994	0.982	0.986	0.981	0.996	0.976	0.986	0.984
		0.7	0.3	0.996	0.973	0.977	0.972	0.974	0.991	0.999	0.981	0.988	0.973	0.998	0.973
		0.7	0.7	0.981	0.981	0.984	0.995	0.990	0.991	0.973	0.979	0.976	0.983	0.978	0.979
Frank	1	0.3	0.3	0.983	0.997	0.989	0.987	0.999	0.985	0.984	0.972	0.989	0.999	0.980	0.984
		0.3	0.7	0.975	0.976	0.974	0.979	0.995	0.981	0.980	0.979	0.991	0.972	0.970	0.997
		0.7	0.3	0.982	1.000	0.995	0.972	0.999	0.978	0.974	0.999	0.974	0.986	0.974	0.992
		0.7	0.7	0.998	0.973	0.973	0.973	0.994	0.993	0.992	0.993	0.982	0.987	0.991	0.976
	2	0.3	0.3	0.997	0.971	0.972	0.972	0.976	0.992	0.988	0.981	0.971	0.994	0.995	0.986
		0.3	0.7	0.998	0.985	0.970	0.982	0.998	0.971	0.998	0.978	0.985	0.976	0.993	0.999
		0.7	0.3	0.979	0.984	0.991	0.990	0.975	0.999	0.970	0.977	0.999	0.973	0.983	0.986
		0.7	0.7	0.983	0.995	0.993	0.973	0.989	0.994	0.999	1.000	0.971	0.986	0.979	0.993
	3	0.3	0.3	0.994	0.988	0.978	0.984	0.978	0.982	0.996	0.978	0.971	0.997	0.975	0.992
		0.3	0.7	0.991	0.990	0.984	0.989	0.976	0.987	0.972	0.973	0.994	0.976	0.985	0.994
		0.7	0.3	0.994	0.977	0.996	1.000	0.978	0.995	0.976	0.996	0.976	0.983	0.995	0.990
		0.7	0.7	0.980	0.978	0.971	0.989	0.979	0.991	0.980	0.988	0.993	0.972	0.987	0.986

				n										
	4	0.3	0.3	0.981	0.971	0.988	0.979	0.990	0.981	0.990	0.987	0.972	0.994	0.982	0.996
		0.3	0.7	0.988	0.987	0.973	0.976	0.980	0.992	0.976	0.990	0.985	0.976	0.992	0.978
		0.7	0.3	0.975	0.979	0.970	0.997	0.975	0.994	0.978	0.990	0.977	0.994	0.971	0.999
		0.7	0.7	0.986	0.985	0.990	0.994	0.979	0.984	0.976	0.998	0.987	0.990	0.996	0.983

Table 2: Testing performance of the Periodic Copula Model to detect relationship between two PARFBM(1) time series ($H = 0.25$), in terms of the Correlation Coefficient (r), Nash-Sutcliff (NSE) and Willmott Index (WI)

Copula	T	θ	β	n											
				120			240			480			1200		
				r	WI	NS	r	WI	NS	r	WI	NS	r	WI	NS
Gaussian	1	0.3	0.3	0.990	0.996	0.981	0.982	1.000	0.980	0.986	0.998	0.989	0.977	0.992	0.994
		0.3	0.7	0.975	0.978	0.981	0.997	0.992	0.990	0.971	0.981	0.996	0.990	0.971	0.978
		0.7	0.3	0.990	0.997	0.984	0.995	0.997	0.973	0.974	0.999	0.992	0.983	0.985	0.974
		0.7	0.7	0.995	0.999	0.983	0.996	0.994	0.974	0.998	0.986	0.972	0.977	0.971	0.996
	2	0.3	0.3	0.985	0.985	0.997	0.992	0.979	0.992	0.999	0.978	0.985	0.999	0.971	0.980
		0.3	0.7	0.971	0.999	0.988	0.995	0.996	0.983	0.993	0.975	0.991	0.970	0.988	0.983
		0.7	0.3	0.999	0.971	0.983	0.979	0.993	0.994	0.984	0.988	0.972	0.976	0.983	0.991
		0.7	0.7	0.994	0.985	0.993	0.971	0.979	0.976	0.971	0.985	0.989	0.996	0.994	0.987
	3	0.3	0.3	0.992	0.993	0.992	0.971	1.000	0.991	0.995	0.985	0.994	0.988	0.982	0.994
		0.3	0.7	0.996	0.997	0.977	0.981	0.991	0.975	0.991	0.980	0.987	0.995	1.000	0.982
		0.7	0.3	0.998	0.976	0.980	0.974	0.974	0.982	0.984	0.979	0.996	0.993	0.971	0.975
		0.7	0.7	0.988	0.985	0.983	0.976	0.993	0.993	0.985	0.985	0.995	0.989	0.994	0.978
	4	0.3	0.3	0.995	0.996	0.990	0.991	0.989	0.996	0.999	0.990	0.978	0.993	0.987	0.980
		0.3	0.7	1.000	0.988	0.990	0.993	0.970	0.972	0.993	0.982	0.977	0.999	0.985	0.972
		0.7	0.3	0.983	0.980	1.000	0.993	0.977	0.972	0.998	0.978	0.998	0.973	0.970	0.993
		0.7	0.7	0.991	0.977	0.985	0.994	0.983	0.973	0.983	0.975	0.990	0.983	0.982	0.990
T	1	0.3	0.3	0.990	0.996	0.981	0.974	0.971	0.998	0.992	0.989	0.982	0.972	0.980	0.974
		0.3	0.7	0.972	0.987	0.990	0.993	0.996	0.998	0.986	0.972	0.994	0.992	0.978	0.975
		0.7	0.3	0.977	0.984	0.997	0.974	0.999	0.996	0.996	0.989	0.986	0.992	0.995	0.980
		0.7	0.7	0.973	0.989	0.974	0.981	0.973	0.982	0.992	0.979	0.985	0.994	0.980	0.973
	2	0.3	0.3	0.996	0.975	0.999	0.976	0.995	0.979	0.978	0.985	0.979	0.984	0.999	0.990
		0.3	0.7	0.977	0.979	0.995	0.995	0.976	0.981	0.973	0.974	0.983	0.989	0.976	0.979
		0.7	0.3	0.996	0.971	0.977	0.983	0.981	0.989	0.972	0.998	0.984	0.980	0.990	0.981
		0.7	0.7	0.994	0.990	0.997	0.997	0.992	0.980	0.987	0.978	0.987	0.992	0.986	0.974
	3	0.3	0.3	0.980	0.986	0.992	0.979	0.973	0.998	0.997	0.988	0.975	0.988	0.981	0.992
		0.3	0.7	0.978	0.999	0.984	0.990	0.986	0.993	0.978	0.980	0.977	0.972	0.979	0.987
		0.7	0.3	0.983	0.986	0.983	0.993	0.986	0.987	0.971	0.992	0.980	0.972	0.985	0.979
		0.7	0.7	0.998	0.988	0.991	0.987	0.984	0.977	0.980	0.993	0.997	0.985	0.989	0.985
	4	0.3	0.3	0.980	1.000	0.994	0.990	0.988	0.982	0.980	0.986	0.997	0.996	0.987	0.973
		0.3	0.7	0.989	0.978	0.984	0.987	0.979	0.976	0.985	0.995	0.989	0.972	0.974	0.986
		0.7	0.3	0.975	0.978	0.982	0.982	0.990	0.993	0.977	0.981	0.972	0.985	0.999	0.993
		0.7	0.7	0.985	0.995	0.974	0.986	0.995	0.993	0.972	0.990	0.994	0.996	0.991	1.000
	1	0.3	0.3	0.986	0.985	0.998	0.980	0.974	0.978	0.992	0.978	1.000	0.971	0.984	0.975
		0.3	0.7	0.986	0.973	0.977	0.984	0.980	0.993	0.986	0.971	0.999	0.979	0.980	0.980
		0.7	0.3	0.995	0.979	0.987	0.975	0.977	0.972	0.981	0.994	0.975	0.982	0.977	0.980
		0.7	0.7	0.995	0.984	0.974	0.973	0.995	0.975	0.987	0.974	0.987	0.993	0.976	0.978
	2	0.3	0.3	0.987	0.999	0.983	0.991	0.974	0.989	0.979	0.972	0.996	0.994	0.996	0.981
		0.3	0.7	0.981	0.980	0.973	0.989	0.975	0.995	0.995	0.976	0.974	0.981	0.980	1.000
		0.7	0.3	0.978	1.000	0.977	0.970	0.994	0.993	0.985	0.971	0.975	0.974	0.978	0.989
		0.7	0.7	0.992	0.980	0.972	0.973	0.974	0.981	0.973	0.980	0.975	0.994	0.986	0.975

Clayton	3	0.3	0.3	0.995	0.987	0.974	0.988	0.989	0.989	0.986	0.976	0.992	0.992	0.991	0.981
		0.3	0.7	0.972	0.987	0.999	0.984	0.988	0.991	0.991	0.974	0.987	0.978	0.981	0.977
		0.7	0.3	0.997	0.985	0.998	0.987	0.978	0.973	0.979	0.972	0.983	0.994	0.977	0.974
		0.7	0.7	0.996	0.989	0.999	0.998	0.980	0.996	0.975	0.970	0.983	0.996	0.976	0.986
	4	0.3	0.3	0.978	0.981	0.987	0.980	0.979	0.988	0.983	0.989	0.974	0.975	0.991	0.992
		0.3	0.7	0.994	0.976	0.993	0.970	0.983	0.999	0.980	0.976	0.986	0.975	0.981	0.992
		0.7	0.3	0.991	0.999	0.977	0.981	0.979	0.992	0.976	0.971	0.989	0.995	0.978	0.971
		0.7	0.7	0.978	0.977	0.981	0.976	0.988	0.989	0.979	0.977	0.979	0.985	0.985	0.988
Gumbel	1	0.3	0.3	0.981	0.986	0.982	0.973	0.971	0.970	0.982	0.999	0.986	0.994	0.983	0.999
		0.3	0.7	0.991	1.000	0.991	0.977	0.980	0.987	0.976	0.996	0.993	0.979	0.990	0.975
		0.7	0.3	0.997	0.971	0.975	0.990	0.986	0.994	0.972	0.983	0.996	0.994	0.972	0.994
		0.7	0.7	0.987	0.988	0.993	0.977	0.993	0.994	0.997	0.995	0.977	0.977	0.981	0.982
	2	0.3	0.3	0.983	0.982	0.983	0.971	0.985	0.979	0.979	0.985	0.972	0.975	1.000	0.977
		0.3	0.7	0.997	0.975	0.970	0.996	0.987	0.976	0.983	0.981	0.982	0.995	0.980	0.973
		0.7	0.3	0.985	0.990	0.974	0.999	0.989	0.996	0.999	0.994	0.989	0.985	0.983	0.971
		0.7	0.7	0.994	0.995	0.999	0.999	0.993	0.981	0.994	0.985	0.970	0.996	0.990	0.994
	3	0.3	0.3	0.972	0.979	0.985	0.974	1.000	0.974	0.976	0.992	0.995	0.981	0.982	0.986
		0.3	0.7	0.975	0.977	0.985	0.971	0.976	0.995	0.992	0.981	0.975	0.976	0.986	0.978
		0.7	0.3	0.973	0.994	0.999	0.992	0.991	0.988	0.997	0.988	0.999	0.986	0.980	0.992
		0.7	0.7	0.982	0.993	0.982	0.999	0.979	0.976	0.997	0.988	0.996	0.987	0.993	0.991
	4	0.3	0.3	0.974	0.999	0.990	0.982	0.994	0.970	0.998	0.988	0.994	0.976	1.000	0.993
		0.3	0.7	0.977	0.986	0.991	0.990	0.997	0.981	0.984	0.997	0.986	0.987	0.972	0.975
		0.7	0.3	0.999	0.976	0.986	0.972	0.976	0.986	0.998	0.995	0.991	0.977	0.971	0.979
		0.7	0.7	0.973	0.996	0.997	0.977	0.988	0.997	0.994	0.990	0.978	0.988	0.988	0.984
Frank	1	0.3	0.3	0.990	0.981	0.979	0.998	0.987	0.993	0.973	0.983	0.999	0.976	0.979	0.989
		0.3	0.7	0.984	0.989	0.990	0.985	0.990	0.999	0.978	0.987	0.980	0.987	0.972	0.982
		0.7	0.3	0.984	0.996	0.971	0.979	0.988	0.998	0.980	0.988	0.982	0.984	0.979	0.991
		0.7	0.7	0.992	0.977	0.986	0.992	0.997	0.996	0.991	0.974	0.972	0.987	0.972	0.996
	2	0.3	0.3	0.986	0.992	0.974	0.989	0.973	0.982	0.997	0.976	0.979	0.981	0.978	0.971
		0.3	0.7	0.993	0.971	0.982	0.977	0.991	0.981	0.980	0.988	0.983	0.991	0.973	0.987
		0.7	0.3	0.973	0.982	0.989	0.985	0.975	0.986	0.983	0.998	0.982	0.976	0.992	0.972
		0.7	0.7	0.984	0.974	0.988	0.980	0.978	0.989	0.981	0.983	0.987	0.997	0.997	0.979
	3	0.3	0.3	0.970	0.987	0.970	0.979	0.995	0.992	0.979	0.994	0.988	0.986	0.999	0.997
		0.3	0.7	0.976	0.995	0.976	0.999	0.986	0.994	0.980	0.996	0.988	0.984	0.982	0.984
		0.7	0.3	0.996	0.992	0.987	0.976	0.979	0.984	0.982	0.990	0.998	0.992	0.993	0.990
		0.7	0.7	0.987	0.990	0.975	0.974	0.998	0.978	0.988	0.972	1.000	0.994	0.987	0.973
	4	0.3	0.3	0.997	0.981	0.994	0.993	0.985	0.979	0.986	0.991	0.985	0.996	0.984	0.987
		0.3	0.7	0.975	0.987	0.980	0.978	0.971	0.987	0.998	0.990	0.997	0.992	0.985	0.993
		0.7	0.3	0.987	0.975	0.973	0.995	0.981	0.973	0.993	0.970	0.978	0.983	0.982	0.995
		0.7	0.7	0.984	0.987	0.998	0.999	0.982	0.996	0.983	0.988	0.996	0.992	0.986	0.991

4. Conclusion

The analysis of hydrological and climatological datasets often requires the detection of relationships between different variables. When we face with the relationship of two time series, cross-correlation function and copula models are suggested. Cross-correlation function is somewhat sensitive to abnormality of datasets and existence of outliers. An efficient way to model dependency is to use new modeling mechanism of Copula Theory which helps understand the correlation beyond linearity. Copula technique is most efficient for stationary time series and may not work well for non-stationary time series such as cyclostationary time series. To solve

this issue, in this research, we introduced a copula-based regression technique. The ability of the proposed approach to detect relationship between two cyclostationary time series with fractional Brownian motion errors was studied. For this purpose, numerous datasets from two cyclostationary time series with fractional Brownian motion errors were produced and analyzed. The results indicated that the values of goodness of fit indices were close to one, and consequently, the proposed method was robust to detect relationship between two cyclostationary time series with fractional Brownian motion errors.

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