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Balcobalancing Numbers

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Abstract: In this work we determine the general terms of balcobalancing numbers, Lucas-balcobalancing numbers and balcobalancers. Further we formulate the sums of these numbers and derive relationship with square triangular numbers.
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1 Introduction

A positive integer n is called a balancing number ([2]) if the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1}$$

holds for some positive integer r which is called balancer corresponding to n. If n is a balancing number with balancer r, then from (1)

$$r=\frac{-2n-1+\sqrt{8n^2+1}}{2}. \tag{2}$$

From (2) we note that n is a balancing number if and only if $8n^2+1$ is a perfect square. Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (2) we note that $8(0)^2+1=1$ and $8(1)^2+1=3^2$ are perfect squares. So we accept 0 and 1 to be balancing numbers. Let B_n denote the n^{th} balancing number. Then $B_0=0, B_1=1$, $B_2=6$ and $B_{n+1}=6B_n-B_{n-1}$ for $n\geq 2$.

Later Panda and Ray ([14]) defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r) \tag{3}$$

holds for some positive integer r which is called cobalancer corresponding to n. If n is a cobalancing number with cobalancer r, then from (3)

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8n + 1}}{2}. \tag{4}$$

From (4) we note that n is a cobalancing number if and only if $8n^2 + 8n + 1$ is a perfect square. Since $8(0)^2 + 8(0) + 1 = 1$ is a perfect square, we accept 0 to be a cobalancing number, just like Behera and Panda accepted $0, 1$ balancing numbers. Cobalancing number is denoted by b_n. Then $b_0 = b_1 = 0, b_2 = 2$ and $b_{n+1} = 6b_n - b_{n-1} + 2$ for $n \geq 2$.

It is clear from (1) and (3) that every balancing number is a cobalancer and every cobalancing number is a balancer, that is, $B_n = r_{n+1}$ and $R_n = b_n$ for $n \geq 1$, where R_n is the n^{th} the balancer and r_n is the n^{th} cobalancer. Since $R_n = b_n$, we get from (1) that

$$b_n = \frac{-2B_n - 1 + \sqrt{8B_n^2 + 1}}{2} \quad \text{and} \quad B_n = \frac{2b_n + 1 + \sqrt{8b_n^2 + 8b_n + 1}}{2}. \tag{5}$$

Thus from (5), we see that $C_n = \sqrt{8B_n^2 + 1}$ and $c_n = \sqrt{8b_n^2 + 8b_n + 1}$ are integers which are called the n^{th} Lucas-balancing number and n^{th} Lucas-cobalancing number, respectively.

Let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ be the roots of the characteristic equation for Pell and Pell-Lucas numbers which are the numbers defined by $P_0 = 0, P_1 = 1$, $P_n = 2P_{n-1} + P_{n-2}$ and $Q_0 = Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$. Then Ray ([17]) derived some nice results on balancing numbers and Pell numbers his Phd thesis. Since x is a balancing number if and only if $8x^2 + 1$ is a perfect square, he set $8x^2 + 1 = y^2$ for some integer $y \geq 1$. Then he get the Pell equation ([1, 3, 9])

$$y^2 - 8x^2 = 1. \tag{6}$$

The fundamental solution of (6) is $(y_1, x_1) = (3, 1)$. So $y_n + x_n\sqrt{8} = (3 + \sqrt{8})^n$ for $n \geq 1$ and similarly $y_n - x_n\sqrt{8} = (3 - \sqrt{8})^n$. Let $\gamma = 3 + \sqrt{8}$ and $\delta = 3 - \sqrt{8}$. Then he get $x_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$ which is the Binet formula for balancing numbers, that is, $B_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$. Since $\alpha^2 = \gamma$ and $\beta^2 = \delta$, he conclude that the Binet formula for balancing numbers is $B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}$. Similarly he get $b_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, C_n = \frac{\alpha^{2n} + \beta^{2n}}{2}$ and $c_n = \frac{\alpha^{2n-1} + \beta^{2n-1}}{2}$ for $n \geq 1$ (see also [10, 13, 16]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [7], Liptai proved that there is no Fibonacci balancing number except 1 and in [8] he proved that there is no Lucas balancing number. In [20], Szalay considered the same problem and obtained some nice results by a different method. In [5], Kovács, Liptai and Olajos extended the concept of balancing numbers to the (a, b)-balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a + b) + \cdots + (a(n - 1) + b) = (a(n + 1) + b) + \cdots + (a(n + r) + b)$$

for some positive integers n and r, then $an + b$ is an (a, b)-balancing number. The sequence of (a, b)-balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [6], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ such that $y \geq 4$. Then a positive integer x with $x \leq y - 2$ is called a (k, l)-power numerical center for y if $1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l$. They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k, l)-power numerical centers. For positive integers k, x, let $\Pi_k(x) = x(x+1)\ldots(x+k-1)$. Then it was proved in [5] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. In [23] Tengely, considered the case $k = 5$, that is, $B_m = x(x+1)(x+2)(x+3)(x+4)$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [15], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers and in [18], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods. In [12], Panda and Panda defined the almost balancing number and its balancer. In [21], the first author considered almost balancing numbers, triangular numbers and square triangular numbers and in [22], he considered the sums and spectral norms of almost balancing numbers.

2 Balcobalancing Numbers.

In this work we define a new balancing number called balcobalancing number, Lucas-balcobalancing number and balcobalancer and determine the general terms of them.

If we sum of both sides of (1) and (3), then we get the Diophantine equation

$$1+2+\cdots+(n-1)+1+2+\cdots+(n-1)+n = 2[(n+1)+(n+2)+\cdots+(n+r)]. \quad (7)$$

So a positive integer n is called a balcobalancing number if the Diophantine equation (7) holds for some positive integer r which is called balcobalancer. For example, $10, 348, 11830, 401880, \cdots$ are balcobalancing numbers with balcobalancers $4, 144, 4900, 166464, \cdots$.

From (7), we get

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 4n + 1}}{2}. \quad (8)$$

Let B_n^{bc} denote the n^{th} balcobalancing number and let R_n^{bc} denote the n^{th} balcobalancer. Then from (8), we get B_n^{bc} is a balcobalancing number if and only if $8(B_n^{bc})^2 + 4B_n^{bc} + 1$ is a perfect square. Thus

$$C_n^{bc} = \sqrt{8(B_n^{bc})^2 + 4B_n^{bc} + 1} \quad (9)$$

is an integer which are called the n^{th} Lucas-balcobancing number. For example $29, 985, 33461, \cdots$ are Lucas-balcobancing numbers (Here we notice that balcobalancing numbers should be grater that 0. But $8(0)^2 + 4(0) + 1 = 1$ is a perfect square, so we assume that 0 is a balcobalancing number, that is, $B_0^{bc} = 0$. In this case, $R_0^{bc} = 0$ and $C_0^{bc} = 1$).

In order to determine the general terms of balcobalancing numbers, Lucas-balcobancing numbers and balcobalancers, we have to determine the set of all (positive) integer solutions of the Pell equation

$$x^2 - 2y^2 = -1. \quad (10)$$

We see from (8) that B_n^{bc} is a balcobalancing number if and only if $8(B_n^{bc})^2+4B_n^{bc}+1$ is a perfect square. So we set $8(B_n^{bc})^2+4B_n^{bc}+1=y^2$ for some integer $y\geq 1$. If we multiply both sides of the last equation by 2, then we get $16(B_n^{bc})^2+8B_n^{bc}+2=2y^2$ and hence $(4B_n^{bc}+1)^2+1=2y^2$. Taking $x=4B_n^{bc}+1$, we get the Pell equation in (10).

For the set of all integer solutions of (10), we can give the following theorem.

Theorem 2.1. *The set of all integer solutions of (10) is* $\{(c_n, 2b_n+1) : n\geq 1\}$.

Proof. For the Pell equation $x^2-2y^2=-1$, the set of representatives is $\text{Rep}=\{[\pm 1 \quad 1]\}$ and $M=\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$. In this case $[-1 \quad 1]M^n$ generates all integer solutions (x_n, y_n) for $n\geq 1$. It can be easily seen that the n^{th} power of M is $M^n=\begin{bmatrix} C_n & 2B_n \\ 4B_n & C_n \end{bmatrix}$ for $n\geq 1$. So

$$[x_n \quad y_n]=[-1 \quad 1]\begin{bmatrix} C_n & 2B_n \\ 4B_n & C_n \end{bmatrix}=[-C_n+4B_n \quad -2B_n+C_n].$$

Thus the set of all integer solutions is $\{(-C_n+4B_n, -2B_n+C_n) : n\geq 1\}$. But we notice that

$$-C_n+4B_n=-(\frac{\alpha^{2n}+\beta^{2n}}{2})+4(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}})=\frac{\alpha^{2n-1}+\beta^{2n-1}}{2}=c_n$$

and similarly $-2B_n+C_n=2b_n+1$. So the result is clear. □

From Theorem 2.1, we can give the following result.

Theorem 2.2. *The general terms of balcobalancing numbers, Lucas-balcobalancing numbers and balcobalancers are*

$$B_n^{bc}=\frac{c_{2n+1}-1}{4},\ C_n^{bc}=2b_{2n+1}+1 \ \textit{and} \ R_n^{bc}=\frac{4b_{2n+1}-c_{2n+1}+1}{4}$$

for $n\geq 1$.

Proof. Notice that we multiply the equation $8(B_n^{bc})^2+4B_n^{bc}+1=y^2$ by 2 and since $x=4B_n^{bc}+1$, we get

$$B_n^{bc}=\frac{x_{2n+1}-1}{4}=\frac{c_{2n+1}-1}{4}$$

for $n\geq 1$. Thus from (9),

$$\begin{aligned} C_n^{bc} &= \sqrt{8(B_n^{bc})^2+4B_n^{bc}+1} \\ &= \sqrt{8(\frac{c_{2n+1}-1}{4})^2+4(\frac{c_{2n+1}-1}{4})+1} \\ &= \sqrt{\frac{c_{2n+1}^2+1}{2}} \\ &= \sqrt{\frac{(\frac{\alpha^{4n+1}+\beta^{4n+1}}{2})^2+1}{2}} \end{aligned}$$

$$= \sqrt{\left[2(\frac{\alpha^{4n+1} - \beta^{4n+1}}{2\sqrt{2}} - \frac{1}{2}) + 1\right]^2}$$
$$= 2b_{2n+1} + 1$$

and from (8), we conclude that

$$R_n^{bc} = \frac{-2(\frac{c_{2n+1}-1}{4}) - 1 + 2b_{2n+1} + 1}{2} = \frac{4b_{2n+1} - c_{2n+1} + 1}{4}.$$

This completes the proof. □

We can also give the general terms of balcobalancing numbers, Lucas-balcobalancing numbers and balcobalancers in terms of balancing and cobalancing numbers as follows.

Theorem 2.3. *The general terms of balcobalancing numbers, Lucas-balcobalancing numbers and balcobalancers are*

$$B_n^{bc} = \frac{B_{2n} + b_{2n+1}}{2}, C_n^{bc} = 2b_{2n+1} + 1 \textit{ and } R_n^{bc} = \frac{-B_{2n} + b_{2n+1}}{2}$$

for $n \geq 1$.

Proof. We proved in Theorem 2.2 that $B_n^{bc} = \frac{c_{2n+1}-1}{4}$. So we easily deduce that

$$\begin{aligned} B_n^{bc} &= \frac{c_{2n+1} - 1}{4} \\ &= \frac{\alpha^{4n+1} + \beta^{4n+1}}{8} - \frac{1}{4} \\ &= \frac{\alpha^{4n+1}(\frac{\alpha^{-1}+1}{4\sqrt{2}}) + \beta^{4n+1}(\frac{-\beta^{-1}-1}{4\sqrt{2}})}{2} - \frac{1}{4} \\ &= \frac{\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}} + \frac{\alpha^{4n+1}-\beta^{4n+1}}{4\sqrt{2}} - \frac{1}{2}}{2} \\ &= \frac{B_{2n} + b_{2n+1}}{2}. \end{aligned}$$

$C_n^{bc} = 2b_{2n+1} + 1$ is already proved in Theorem 2.2. Similarly we get $R_n^{bc} = \frac{-B_{2n}+b_{2n+1}}{2}$. □

Recall that the general terms of all balancing numbers can be given in terms of Pell numbers, namely,

$$B_n = \frac{P_{2n}}{2}, b_n = \frac{P_{2n-1} - 1}{2}, C_n = P_{2n} + P_{2n-1}, c_n = P_{2n-1} + P_{2n-2}.$$

Similarly we can give the following theorem.

Theorem 2.4. *The general terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers are*

$$B_n^{bc} = \frac{P_{4n+1} + P_{4n} - 1}{4}, C_n^{bc} = P_{4n+1} \textit{ and } R_n^{bc} = \frac{P_{4n+1} - P_{4n} - 1}{4}$$

for $n \geq 1$.

Proof. Notice that

$$c_{2n+1} = \frac{\alpha^{4n+1} + \beta^{4n+1}}{2} = \frac{\alpha^{4n+1}(1+\alpha^{-1}) - \beta^{4n+1}(1+\beta^{-1})}{2\sqrt{2}} = P_{4n+1} + P_{4n}.$$

So from Theorem 2.2, we get

$$B_n^{bc} = \frac{c_{2n+1} - 1}{4} = \frac{P_{4n+1} + P_{4n} - 1}{4}.$$

The others can be proved similarly. □

We notice that n is a a balancing number if and only if n^2 is a triangular number (triangular numbers denoted by T_n are the numbers of the form $T_n = \frac{n(n+1)}{2}$ for $n \geq 0$). Indeed from (1) we get $\frac{(n+r)(n+r+1)}{2} = n^2$, that is,

$$T_{B_n + R_n} = B_n^2.$$

Similarly we can give the following theorem.

Theorem 2.5. *B_n^{bc} is a balcobalancing number if and only if $(B_n^{bc})^2 + \frac{B_n^{bc}}{2}$ is a triangular number, that is,*

$$T_{B_n^{bc} + R_n^{bc}} = (B_n^{bc})^2 + \frac{B_n^{bc}}{2}.$$

Proof. From (7), we get $n^2 = 2nr + r(r+1)$ and hence

$$\frac{(n+r)(n+r+1)}{2} = n^2 + \frac{n}{2}.$$

So the result is obvious. □

As in Theorem 2.5, we can give the following result.

Theorem 2.6. *B_n^{bc} is a balcobalancing number if and only if $(R_n^{bc})^2 + 2B_n^{bc}R_n^{bc} + R_n^{bc} + \frac{B_n^{bc}}{2}$ is a triangular number, that is,*

$$T_{B_n^{bc} + R_n^{bc}} = (R_n^{bc})^2 + 2B_n^{bc}R_n^{bc} + R_n^{bc} + \frac{B_n^{bc}}{2}.$$

Proof. Since $n^2 = 2nr + r(r+1) \Leftrightarrow \frac{(n+r)(n+r+1)}{2} = r^2 + 2nr + r + \frac{n}{2}$, the result is clear. □

We notice that the sum of n^{th} balancing number and it is balancer is equal to the half of the n^{th} Lucas-balancing number -1, that is, $B_n + R_n = \frac{C_n - 1}{2}$. Similarly we can give the following result.

Theorem 2.7. *The sum of n^{th} balcobalancing number and it is balancer is equals to the $(2n+1)^{st}$ cobalancing number, that is, $B_n^{bc} + R_n^{bc} = b_{2n+1}$, and the difference of n^{th} balcobalancing number and it is balancer is equals to the $(2n)^{nd}$ balancing number, that is, $B_n^{bc} - R_n^{bc} = B_{2n}$.*

Proof. From Theorem 2.3, we get the desired result. □

Theorem 2.8. *R_n^{bc} is a perfect square for every $n \geq 1$.*

Proof. Notice that $R_n^{bc} = \frac{-B_{2n}+b_{2n+1}}{2}$ by Theorem 2.3. So we get

$$\begin{aligned}\sqrt{R_n^{bc}} &= \sqrt{\frac{-B_{2n}+b_{2n+1}}{2}} \\ &= \sqrt{\frac{-\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}} + \frac{\alpha^{4n+1}-\beta^{4n+1}}{4\sqrt{2}} - \frac{1}{2}}{2}} \\ &= \sqrt{\frac{\alpha^{4n}+\beta^{4n}-2}{8}} \\ &= \sqrt{\left[2(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}})\right]^2} \\ &= 2B_n\end{aligned}$$

as we wanted. □

Further we can give the following theorem.

Theorem 2.9. *The ration of the n^{th} balcobalancing number to the n^{th} balancing number is*

$$\frac{B_n^{bc}}{B_n} = 4b_{n+1} + 2$$

and the ration of the n^{th} Lucas-balcobalancing number to the n^{th} Lucas-balancing number is

$$\frac{C_n^{bc}}{C_n} = \frac{2b_{2n+1}+1}{2B_n + 2b_n + 1}$$

for $n \geq 1$. The ration of the n^{th} balcobalancer to the n^{th} balancer is

$$\frac{R_n^{bc}}{R_n} = \begin{cases} \frac{8C_{\frac{n}{2}}^2 B_{\frac{n}{2}}}{c_{\frac{n}{2}}} & \text{for even } n \geq 2 \\ \frac{2(2b_{\frac{n+1}{2}}+1)^2 c_{\frac{n+1}{2}}}{B_{\frac{n-1}{2}}} & \text{for odd } n \geq 3. \end{cases}$$

Proof. It can be easily derived from Theorem 2.3. □

3 Binet Formulas, Recurrence Relations and Companion Matrix.

Theorem 3.1. *The Binet formulas for balcobalancing numbers, Lucas-balcobalancing numbers and balcobalancers are*

$$B_n^{bc} = \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4},\ C_n^{bc} = \frac{\alpha^{4n+1}-\beta^{4n+1}}{2\sqrt{2}} \text{ and } R_n^{bc} = \frac{\alpha^{4n}+\beta^{4n}}{8} - \frac{1}{4}$$

for $n \geq 1$.

Proof. Note that $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$ and $b_n = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$. So we get from Theorem 2.3 that

$$\begin{aligned} B_n^{bc} &= \frac{B_{2n}+b_{2n+1}}{2} \\ &= \frac{\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}} + \frac{\alpha^{4n+1}-\beta^{4n+1}}{4\sqrt{2}} - \frac{1}{2}}{2} \\ &= \frac{\alpha^{4n}(\frac{1+\alpha}{4\sqrt{2}}) + \beta^{4n}(\frac{-1-\beta}{4\sqrt{2}})}{2} - \frac{1}{4} \\ &= \frac{\alpha^{4n}(\frac{\sqrt{2}(1+\sqrt{2})}{4\sqrt{2}}) + \beta^{4n}(\frac{\sqrt{2}(1-\sqrt{2})}{4\sqrt{2}})}{2} - \frac{1}{4} \\ &= \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}. \end{aligned}$$

The others can be proved similarly. □

Theorem 3.2. *B_n^{bc}, C_n^{bc} and R_n^{bc} satisfy the recurrence relations*

$$\begin{aligned} B_n^{bc} &= 35(B_{n-1}^{bc} - B_{n-2}^{bc}) + B_{n-3}^{bc} \\ R_n^{bc} &= 35(R_{n-1}^{bc} - R_{n-2}^{bc}) + R_{n-3}^{bc} \end{aligned}$$

for $n \geq 3$ and

$$C_n^{bc} = 34C_{n-1}^{bc} - C_{n-2}^{bc}$$

for $n \geq 2$.

Proof. Recall that $B_n^{bc} = \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}$ by Theorem 3.1. Since $35\alpha^{-3} - 35\alpha^{-7} + \alpha^{-11} = \alpha$ and $35\beta^{-3} - 35\beta^{-7} + \beta^{-11} = \beta$, we conclude that

$$\begin{aligned} &35(B_{n-1}^{bc} - B_{n-2}^{bc}) + B_{n-3}^{bc} \\ &= 35\left[(\frac{\alpha^{4n-3}+\beta^{4n-3}}{8} - \frac{1}{4}) - (\frac{\alpha^{4n-7}+\beta^{4n-7}}{8} - \frac{1}{4})\right] + \frac{\alpha^{4n-11}+\beta^{4n-11}}{8} - \frac{1}{4} \\ &= \frac{\alpha^{4n}(35\alpha^{-3} - 35\alpha^{-7} + \alpha^{-11}) + \beta^{4n}(35\beta^{-3} - 35\beta^{-7} + \beta^{-11})}{8} - \frac{1}{4} \\ &= \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4} \\ &= B_n^{bc} \end{aligned}$$

The others can be proved similarly. □

Recall that the companion matrix for balancing numbers is

$$M = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}.$$

It can be easily seen that the n^{th} power of M is

$$M^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}$$

for $n \geq 1$. Since $B_n^{bc} = 35(B_{n-1}^{bc} - B_{n-2}^{bc}) + B_{n-3}^{bc}$ and $R_n^{bc} = 35(R_{n-1}^{bc} - R_{n-2}^{bc}) + R_{n-3}^{bc}$ by Theorem 3.2, the companion matrix for balcobalancing numbers and balcobalancers are same and is

$$M^{bc} = \begin{bmatrix} 35 & -35 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and since $C_n^{bc} = 34C_{n-1}^{bc} - C_{n-2}^{bc}$, the companion matrix for Lucas-balcobalancing numbers is

$$N^{bc} = \begin{bmatrix} 34 & -1 \\ 1 & 0 \end{bmatrix}.$$

Hence we can give the following theorem.

Theorem 3.3. *The n^{th} power of M^{bc} is*

$$(M^{bc})^n = \begin{bmatrix} \sum\limits_{i=0}^{\frac{n}{2}} B_{4i+1} & -\sum\limits_{i=1}^{n} B_{2i+1} & \sum\limits_{i=0}^{\frac{n-2}{2}} B_{4i+3} \\ \sum\limits_{i=0}^{\frac{n-2}{2}} B_{4i+3} & -\sum\limits_{i=1}^{n-1} B_{2i+1} & \sum\limits_{i=0}^{\frac{n-2}{2}} B_{4i+1} \\ \sum\limits_{i=0}^{\frac{n-2}{2}} B_{4i+1} & -\sum\limits_{i=1}^{n-2} B_{2i+1} & \sum\limits_{i=0}^{\frac{n-4}{2}} B_{4i+3} \end{bmatrix}$$

for even $n \geq 4$ or

$$(M^{bc})^n = \begin{bmatrix} \sum\limits_{i=0}^{\frac{n-1}{2}} B_{4i+3} & -\sum\limits_{i=1}^{n} B_{2i+1} & \sum\limits_{i=0}^{\frac{n-1}{2}} B_{4i+1} \\ \sum\limits_{i=0}^{\frac{n-1}{2}} B_{4i+1} & -\sum\limits_{i=1}^{n-1} B_{2i+1} & \sum\limits_{i=0}^{\frac{n-3}{2}} B_{4i+3} \\ \sum\limits_{i=0}^{\frac{n-3}{2}} B_{4i+3} & -\sum\limits_{i=1}^{n-2} B_{2i+1} & \sum\limits_{i=0}^{\frac{n-3}{2}} B_{4i+1} \end{bmatrix}$$

for odd $n \geq 3$, and the n^{th} power of N^{bc} is

$$(N^{bc})^n = (-1)^n \begin{bmatrix} \sum\limits_{i=1}^{n+1} (-1)^{i+1} B_{2i-1} & \sum\limits_{i=1}^{n} (-1)^{i+1} B_{2i-1} \\ -\sum\limits_{i=1}^{n} (-1)^{i+1} B_{2i-1} & -\sum\limits_{i=1}^{n-1} (-1)^{i+1} B_{2i-1} \end{bmatrix}$$

for every $n \geq 1$.

Proof. It can be proved by induction on n. □

We can rewrite the n^{th} power of M^{bc} and N^{bc} in terms of balancing and Lucas-balancing numbers instead of sums of balancing numbers. For this purpose, we set the integer sequences

$$k_n = \frac{-8B_{2n} + 3C_{2n} - 3}{96} \text{ and } l_n = \frac{-288B_{2n} - 102C_{2n} + 102}{96}$$

for $n \geq 0$. Then we can give the following theorem.

Theorem 3.4. *The n^{th} power of M^{bc} is*

$$(M^{bc})^n = \begin{bmatrix} k_{n+2} & l_n & k_{n+1} \\ k_{n+1} & l_{n-1} & k_n \\ k_n & l_{n-2} & k_{n-1} \end{bmatrix}$$

for every $n \geq 2$, and the n^{th} power of N is

$$(N^{bc})^n = (-1)^n \begin{cases} \begin{bmatrix} k_{n+2} - k_{n+1} & k_n - k_{n+1} \\ -k_n + k_{n+1} & -k_n + k_{n-1} \end{bmatrix} & \textit{for even } n \geq 2 \\ \begin{bmatrix} k_{n+1} - k_{n+2} & k_{n+1} - k_n \\ -k_{n+1} + k_n & -k_{n-1} + k_n \end{bmatrix} & \textit{for odd } n \geq 1. \end{cases}$$

Proof. It can be proved by induction on n. □

4 Sums of Balcobalancing Numbers.

Theorem 4.1. *The sum of first $n-$terms of B_n^{bc}, C_n^{bc} and R_n^{bc} is*

$$\sum_{i=1}^{n} B_i^{bc} = \frac{b_{2n+2} - 2n - 2}{8}$$
$$\sum_{i=1}^{n} C_i^{bc} = \frac{c_{2n+2} - 7}{8}$$
$$\sum_{i=1}^{n} R_i^{bc} = \frac{B_{2n+1} - 2n - 1}{8}$$

for $n \geq 1$.

Proof. Recall that $B_n^{bc} = \frac{\alpha^{4n+1} + \beta^{4n+1}}{8} - \frac{1}{4}$ by Theorem 3.1. So

$$\sum_{i=1}^{n} B_i^{bc} = \sum_{i=1}^{n} (\frac{\alpha^{4i+1} + \beta^{4i+1}}{8} - \frac{1}{4}). \tag{11}$$

Here we notice that $\sum\limits_{i=1}^{n}\alpha^{4i+1}=\frac{-\alpha^3(1-\alpha^{4n})}{4\sqrt{2}}$ and $\sum\limits_{i=1}^{n}\beta^{4i+1}=\frac{\beta^3(1-\beta^{4n})}{4\sqrt{2}}$. So (11) becomes

$$\begin{aligned}
\sum_{i=1}^{n} B_i^{bc} &= \sum_{i=1}^{n}(\frac{\alpha^{4i+1}+\beta^{4i+1}}{8}-\frac{1}{4}) \\
&= \frac{\frac{-\alpha^3(1-\alpha^{4n})}{4\sqrt{2}}+\frac{\beta^3(1-\beta^{4n})}{4\sqrt{2}}}{8}-\frac{n}{4} \\
&= \frac{\alpha^{4n+3}-\beta^{4n+3}-\alpha^3+\beta^3}{32\sqrt{2}}-\frac{n}{4} \\
&= \frac{\alpha^{4n+3}-\beta^{4n+3}}{32\sqrt{2}}-\frac{5}{16}-\frac{n}{4} \\
&= \frac{(\frac{\alpha^{4n+3}-\beta^{4n+3}}{4\sqrt{2}}-\frac{1}{2})+\frac{1}{2}}{8}-\frac{5}{16}-\frac{n}{4} \\
&= \frac{b_{2n+2}-2n-2}{8}.
\end{aligned}$$

The others can be proved similarly. □

In [13], Panda and Ray proved that the sum of first $2n-1$ Pell numbers is equals to the sum of n^{th} balancing number and its balancer, that is,

$$\sum_{i=1}^{2n-1} P_i = B_n + R_n. \tag{12}$$

Later in [4], Gözeri, Özkoç and Tekcan proved that the sum of Pell-Lucas numbers from 0 to $2n-1$ is equals to the sum of the n^{th} Lucas-balancing and the n^{th} Lucas-cobalancing number, that is,

$$\sum_{i=0}^{2n-1} Q_i = C_n + c_n.$$

As in (12), we can give the following theorem.

Theorem 4.2. *The sum of even ordered Pell numbers from* 1 *to* $(2n)$ *is equals to the sum of the* n^{th} *balcobalancing numbers and its balancer, that is,*

$$\sum_{i=1}^{2n} P_{2i} = B_n^{bc} + R_n^{bc}.$$

Proof. Recall that $P_n=\frac{\alpha^n-\beta^n}{2\sqrt{2}}$. Since $\sum\limits_{i=1}^{2n}\alpha^{2i}=\frac{-\alpha(1-\alpha^{4n})}{2}$ and $\sum\limits_{i=1}^{2n}\beta^{2i}=\frac{-\beta(1-\beta^{4n})}{2}$, we observe that

$$\begin{aligned}
\sum_{i=1}^{2n} P_{2i} &= \sum_{i=1}^{2n}(\frac{\alpha^{2i}-\beta^{2i}}{2\sqrt{2}}) \\
&= \frac{\frac{-\alpha(1-\alpha^{4n})}{2}-\frac{-\beta(1-\beta^{4n})}{2}}{2\sqrt{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha^{4n+1} - \beta^{4n+1}}{4\sqrt{2}} - \frac{1}{2} \\
&= \frac{\alpha^{4n+1}(1+\alpha^{-1}) + \beta^{4n+1}(1+\beta^{-1})}{8} - \frac{1}{2} \\
&= \frac{\alpha^{4n+1} + \beta^{4n+1}}{8} - \frac{1}{4} + \frac{\alpha^{4n} + \beta^{4n}}{8} - \frac{1}{4} \\
&= B_n^{bc} + R_n^{bc}
\end{aligned}
$$

as we claimed. □

Similarly we can give the following theorem which can be proved similarly.

Theorem 4.3. *For the sums of Pell, Pell-Lucas and balancing numbers, we have*

1. *the sum of odd ordered Pell numbers from* 1 *to* $(2n)$ *is equals to the difference of the* n^{th} *balcobalancing number and its balancer, that is,*

$$\sum_{i=1}^{2n} P_{2i-1} = B_n^{bc} - R_n^{bc}.$$

2. *the half of the sum of Pell numbers from* 1 *to* $(4n)$ *is equals to the* n^{th} *balcobalancing number, that is,*

$$\frac{\sum_{i=1}^{4n} P_i}{2} = B_n^{bc}.$$

3. *the sum of Pell-Lucas numbers from* 0 *to* $(4n+1)$ *is equals to the sum of the twelve times of the* n^{th} *balcobalancing number, four times of the its balancer plus* 4*, that is,*

$$\sum_{i=0}^{4n+1} Q_i = 12B_n^{bc} + 4R_n^{bc} + 4.$$

4. *the sum of Pell-Lucas numbers from* 1 *to* $(4n)$ *is equals to the two times of the* n^{th} *Lucas-balcobalancing number mines* 1*, that is,*

$$\sum_{i=1}^{4n} Q_i = 2(C_n^{bc} - 1).$$

5. *the sum of balancing numbers from* 1 *to* $(4n+1)$ *is equals to the product of the three times of the* n^{th} *balcobalancing number, its balancer plus* 1 *and the four times of the* n^{th} *balcobalancing number plus* 1*, that is,*

$$\sum_{i=1}^{4n+1} B_i = (3B_n^{bc} + R_n^{bc} + 1)(4B_n^{bc} + 1).$$

In [19], Santana and Diaz-Barrero proved that the sum of first nonzero $4n+1$ terms of Pell numbers is a perfect square, that is,

$$\sum_{i=1}^{4n+1} P_i = \left[\sum_{i=0}^{n} \begin{pmatrix} 2n+1 \\ 2i \end{pmatrix} 2^i\right]^2.$$

In fact this sum equals to the square of the $(n+1)^{\text{st}}$ Lucas-cobalancing number, that is,

$$\sum_{i=1}^{4n+1} P_i = c_{n+1}^2.$$

Similarly we can give the following result.

Theorem 4.4. *The sum of Pell numbers from* 1 *to* $(8n+1)$ *is a perfect square and is*

$$\sum_{i=1}^{8n+1} P_i = (4B_n^{bc}+1)^2.$$

Proof. It can be proved as in the same way that Theorems 4.1 and 4.2 were proved. □

Also they proved that

$$P_{2n+1} \left| \sum_{i=0}^{2n} P_{2i+1} \right. \quad \text{and} \quad P_{2n} \left| \sum_{i=1}^{2n} P_{2i-1} \right. .$$

Similarly we can give the following result.

Theorem 4.5. $C_n^{bc} \left| \sum_{i=0}^{4n} P_{2i+1} \right.$.

Proof. It can be easily derived that

$$\sum_{i=0}^{4n} P_{2i+1} = C_n^{bc}(4B_n^{bc}+1).$$

So the result is obvious. □

Apart from Theorem 4.4, we can give the following theorem which can be proved similarly.

Theorem 4.6. *For the sums of Pell, Pell-Lucas, balancing and Lucas-cobalancing numbers, we have*

1. *the sum of Pell numbers from* 1 *to* $(8n+3)$ *plus* 1 *is a perfect square and is*

$$1+\sum_{i=1}^{8n+3} P_i = (4B_n^{bc}+2C_n^{bc}+1)^2.$$

2. *the sum of odd ordered Pell-Lucas numbers from* 1 *to* $(4n+2)$ *is a perfect square and is*

$$\sum_{i=1}^{4n+2} Q_{2i-1} = (8B_n^{bc}+2C_n^{bc}+2)^2.$$

3. *the half of the sum of odd Pell-Lucas numbers from* 0 *to* $(4n)$ *is a perfect square and is*

$$\frac{\sum\limits_{i=0}^{4n} Q_{2i+1}}{2} = (4B_n^{bc}+1)^2.$$

4. *the sum of odd ordered balancing numbers from* 1 *to* $(2n+1)$ *is a perfect square and is*

$$\sum_{i=1}^{2n+1} B_{2i-1} = (3B_n^{bc}+R_n^{bc}+1)^2$$

and the four times of the sum of odd ordered balancing numbers from 1 *to* n *is a perfect square and is*

$$4\sum_{i=1}^{n} B_{2i-1} = R_n^{bc} \quad \text{(by Theorem 2.8)}$$

5. *the sum of Lucas-cobalancing numbers from* 1 *to* $(4n+2)$ *plus* 1 *is a perfect square and is*

$$1+\sum_{i=1}^{4n+2} c_i = (8B_n^{bc}+4R_n^{bc}+3)^2.$$

5 Relationship with Square Triangular Numbers.

Recall that there are infinitely many triangular numbers that are also square numbers which are called square triangular numbers and are denoted by S_n. For example, $1, 36, 1225, 41616, \ldots$ are square triangular numbers.

For the n^{th} square triangular number S_n, we can write

$$S_n = s_n^2 = \frac{t_n(t_n+1)}{2},$$

where s_n and t_n are the sides of the corresponding square and triangle. Their Binet formulas are

$$S_n = \frac{\alpha^{4n}+\beta^{4n}-2}{32}, s_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}} \text{ and } t_n = \frac{\alpha^{2n}+\beta^{2n}-2}{4} \tag{13}$$

for $n \geq 1$ (see [2, 11]).

In [21], the first author gave the general terms of almost balancing numbers in terms of square triangular numbers. Similarly, we can give the general terms of balcobalancing numbers, Lucas-balcobalancing numbers and balcobalancers in terms of squares and triangles as follows.

Theorem 5.1. *The general terms of balcobalancing numbers, Lucas-balcobalancing numbers and balcobalancers are*

$$\begin{aligned} B_n^{bc} &= \frac{2s_{2n+1}-t_{2n+1}-1}{2} \\ C_n^{bc} &= -2s_{2n+1}+2t_{2n+1}+1 \\ R_n^{bc} &= \frac{-4s_{2n+1}+3t_{2n+1}+1}{2} \end{aligned}$$

for $n \geq 1$.

Proof. Since $B_n^{bc} = \frac{B_{2n}+b_{2n+1}}{2}$ by Theorem 2.3, we find that

$$\begin{aligned}
B_n^{bc} &= \frac{B_{2n}+b_{2n+1}}{2} \\
&= \frac{\frac{\alpha^{4n}-\beta^{2n}}{4\sqrt{2}} + \frac{\alpha^{4n+1}-\beta^{4n+1}}{4\sqrt{2}} - \frac{1}{2}}{2} \\
&= \frac{\alpha^{4n+1}+\beta^{4n+1}-2}{8} \\
&= \frac{\alpha^{4n+2}(\frac{1}{2\sqrt{2}}-\frac{1}{4}) + \beta^{4n+2}(-\frac{1}{2\sqrt{2}}-\frac{1}{4}) - \frac{1}{2}}{2} \\
&= \frac{2(\frac{\alpha^{4n+2}-\beta^{4n+2}}{4\sqrt{2}}) - (\frac{\alpha^{4n+2}+\beta^{4n+2}-2}{4}) - 1}{2} \\
&= \frac{2s_{2n+1}-t_{2n+1}-1}{2}
\end{aligned}$$

by (13). The others can be proved similarly. □

Finally we can give the following result.

Theorem 5.2. $S_n = \frac{R_n^{bc}}{4}$ *for* $n \geq 1$.

Proof. Appyling (13), we get

$$S_n = \frac{\alpha^{4n}+\beta^{4n}-2}{32} = \frac{\frac{\alpha^{4n}+\beta^{4n}}{8}-\frac{1}{4}}{4} = \frac{R_n^{bc}}{4}$$

by Theorem 3.1. □

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