Some Remarks on Skula Spaces

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This lecture is a survey of a joint work with Taras Banack and Wiesław Kubiś entitled

VIETORIS HYPERSPACES OF SCATTERED PRIESTLEY SPACES [2]: http://arxiv.org/abs/2007.12890

This paper is the continuous of the work well-generated Boolean algebras, in a topological way, started in [3].

Stone duality. If X is a compact and 0-dimensional space then the set $\mathrm{Clop}(X)$ of closed and open (clopen) subsets of X is a Boolean algebra generating the topology of X. Conversely any Boolean algebra B is the algebra of clopen subsets of the compact and 0-dimensional space $\mathrm{Ult}(B) \subseteq \{0,1\}^B$. By duality, we have the following result.

Theorem 1. (§2.3 in [3]) *The space* $\mathrm{Ult}(B)$ *of a Boolean algebra* B *is Skula if and only if* B *is well-generated. That is, by the definition:* B *has a well-founded sublattice generating* B. □

1. Skula spaces.

For a topological space X, we say that a family $\mathcal{U} := \{U_x : x \in X\}$ is a **clopen selector** if each U_x is a closed and open (clopen) subset of X and if $\mathcal{U}$ satisfies:

(1) $x \in U_x$ for every $x \in X$ and

(2) the relation " $x <^{\mathcal{U}} y$ if and only if $x \neq y$ and $x \in U_y$ " is irreflexive and transitive. ■

Therefore a clopen selector $\mathcal{U} := \{U_x : x \in X\}$ for X induces a partial order relation $\leq^{\mathcal{U}}$ on X, defined by

$$x \leq^{\mathcal{U}} y \quad \text{if and only if} \quad U_x \subseteq U_y .$$

Hence $U_x := \{y \in X : y \leq^{\mathcal{U}} x\}$ (also denoted by $\downarrow x$) is a clopen principal ideal of X for any $x \in X$ for the order $\leq^{\mathcal{U}}$.

Remark. The set of U_x's and their complements generate the topology whenever X is compact.

A space X is **Skula** if X is a Hausdorff compact space and has a clopen selector. ■

Theorem 2. [2] *Let* $\mathcal{U} := \{U_x : x \in X\}$ *be a clopen selector for a Skula space* X*. Then*

- *Every (nonempty) closed initial subset of* X *is a finite union of* U_z*'s (notice that* X *and the* U_t*'s are compact clopen sets).*
  In particular for distinct U_x *and* U_y *in* $\mathcal{U}$*,* $U_x \cap U_y$ *is a finite union of* U_z*'s.*

(1) $\langle \mathcal{U}, \subseteq \rangle$ *is well-founded. Therefore* $\langle X, \subseteq \rangle$ *has a well-founded rank:* $\mathrm{rkWF}_X(x) = \sup\{\mathrm{rkWF}_X(y) : y < x\}$. *So* $\mathrm{rkWF}_X(x) = 0$ *if and only if* x *is minimal, i.e.* $U_x = \{x\}$*. Moreover* $\mathrm{rkWF}(X) := \sup_{x \in X} \mathrm{rkWF}_X(x)$.

(2) X *is scattered, i.e. every nonempty subset of* $\mathrm{Ult}(B)$ *has an isolated point (for the induced topology). Therefore we can define the Cantor-Bendixson height (*htCB_X*) of* $x \in X$*. For instance* $\mathrm{htCB}_X(x) = 0$ *if and only if* x *is isolated in* X*. Moreover* $\mathrm{htCB}(X) := \sup_{x \in X} \mathrm{htCB}_X(x)$. □

Since $U_x = \downarrow x := \{y \in X : y \leq x\}$ is an initial and clopen subset of X, we have
$\mathrm{htcb}(U_x) = \mathrm{htcb}_X(x) \leq \mathrm{rkwf}_X(x) = \mathrm{rkwf}(U_x)$ for any $x \in X$, and so $\mathrm{htcb}(X) \leq \mathrm{rkwf}(X)$.

To a Skula space X we can associate its Vietoris hyperspace $H(X)$, that is a "free join-semilattice over X in the category of continuous join semilattice spaces".

We define the **Vietoris hyperspace** $H(X)$ over X as follows:

- $H(X)$ is the set of all nonempty closed initial subsets of $\langle X, \leq \rangle$.
- For $F, G \in H(X)$, we set $F \leq G$ if and only if $F \subseteq G$.
- The topology $\mathcal{T}$ on $H(X)$ is the topology generated by the sets
$$U^+ := \{K \in H(X) : K \subseteq U\} \quad \text{and} \quad V^- := \{K \in H(X) : K \cap V \neq \emptyset\}$$
where U and V are any clopen initial subsets and clopen final subsets in X, respectively. ■

Theorem 3. [2] *Let X be a Skula space. Then $H(X)$ is a Skula space and*

- *$\langle A, B \rangle \mapsto A \vee B := A \cup B$ is a continuous semilattice operation on $H(X)$.*
- *X is topologically embeddable in $H(X)$ by the increasing continuous map $\eta : x \mapsto \downarrow x := U_x$.*
- *The join semilattice generated by $\eta[X]$ in $H(X)$ is topologically dense in $H(X)$.* □

Theorem 4. [2] *Let X be a Skula space and let $\mathcal{U}$ be a clopen selector for X. Then*
$$\mathrm{htcb}(X) \leq \mathrm{rkwf}(X) < \omega^{\mathrm{htCB}(X)+1} \quad \textit{and} \quad \mathrm{rkwf}(H(X)) \leq \omega^{\mathrm{rkWF}(X)}.$$ □

2. Canonical Skula spaces.

A space X is a **canonical Skula space** if X has a clopen selector $\mathcal{U} := \{U_x : x \in X\}$ satisfying one of the following equivalent properties for each $U_x \in \mathcal{U}$:

(i) There is an ordinal α such that the α^{th}–Cantor-Bendixson derivative $D^\alpha(U_x)$ of U_x is the singleton $\{x\}$.

(ii) $\mathrm{rkwf}(U_x) = \mathrm{htcb}(U_x)$ and U_x is unitary (meaning that $D^\beta(U_x)$ is a singleton for some β). ■

Examples. Every contiuous image of a compact ordinal space $\alpha + 1$ (with the order topology) is canonically Skula. The class of canonically Skula spaces is closed under finite product.

Theorem 5. [2] *Let X be a canonical Skula space. Then $H(X)$ is a canonical Skula space.* □

Moreover we can compute $\mathrm{htcb}_{H(X)}(V) = \mathrm{rkwf}_{H(X)}(V)$ for every $V \in H(X)$.

Remark. (1) There is a compact and 0-dimensional space which is not Skula [3].
(2) There is a Skula space which is not canonically Skula [3].

3. Poset spaces.

For a partially ordered set (poset) P we denote by $\mathrm{IS}(P)$ the set of initial subsets of P endowed with the pointwise topology. So $\mathrm{IS}(P)$, as compact subspace of $\{0,1\}^P$, is compact and 0-dimensional, and we can see $H(P) := \mathrm{IS}(P)$ as the "Vietoris hyperspace" of the poset P.

Proposition 6. [1, Theorems 1.3] *Let P be a poset. The space* $\mathrm{IS}(P)$ *is Skula if and only if*

(1) *P is a narrow, i.e. any antichain is finite, and*
(2) *P is order-scattered, i.e. does not contains a copy of the rationals chain $\mathbb{Q}$.* □

Recall that a well-quasi ordering (wqo) is a narrow and well-founded poset. From the above result, M. Pouzet asks for the following question.

Question (M. Pouzet)**.** Let P be a well-quasi ordering. Is $\mathrm{IS}(P)$ canonically Skula?

We do not know the answer of this question even if P is covered by finitely many well-orderings.

References

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[3] R. Bonnet, M. Rubin: *On well-generated Boolean algebras*, Ann. Pure Appl. Logic **105** (2000), 1–50.