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*Analytical resolution on Squeezing Unsteady nanofluid flow problem between two parallel plates by the Homotopic Perturbation Method.*

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**Abstract**

Heat transfer behavior of unsteady flow of squeezing nanofluid (Copper+water) between two parallel plates is investigated. By using the appropriate transformation for the velocity and temperature, the basic equations governing the flow and heat transfer were reduced to a set of ordinary differential equations. These equations subjected to the associated boundary conditions were solved analytically using Homotopy Perturbation Method and numerically using Runge-Kutta-Fehlberg method with shooting technique. Effects on the behavior of velocity and temperature for various values of relevant parameters are illustrated graphically. The skin-friction coefficient, heat transfer and Nusselt number rate are also tabulated for various governing parameters. The results indicate that, for nanofluid flow, the rates of heat transfer and velocity had direct relationship with squeeze number and nanoparticle volume fraction they are also a decreasing function of those parameters.

**Keywords** : Nanofluid; Homotopy perturbation method; Runge-Kutta-Fehlberg;

**1. Introduction**

The word nanofluid represents the fluid in which particles of size with order of nanometer (diameter < 100 nm) are mixed in the base fluid. The nanoparticles used in nanofluids are generally made of metals (Al, Cu), oxides (Al<sub>2</sub>O<sub>3</sub>, CuO, TiO<sub>2</sub>, and SiO<sub>2</sub>), carbides (SiC), nitrides (AlN, SiN), and nonmetal (graphite, carbon nanotubes) and the base fluid is usually a conductive fluid, such as water or ethylene glycol. Other base fluids are toluene, oil, other lubricants, biofluids, and polymer solution. Nanoparticles are present up to 5% volume fraction in nanofluids. The conventional heat transfer fluids are poor conductors of heat. Nanofluids make an edge over them because they have high heat transfer capability. Since these heating/cooling fluids play a vital role in the development of energy efficient heat transfer equipment for energy supply, to raise the thermal conductivity of these fluids, nanosized conducting metal particles are added to them. Therefore, their proper

understanding is a must to use them efficiently in modern industry. Applications of nanofluids include microelectronics, fuel cells, and pharmaceutical processes.

Choi and Eastman [1] were the first to propose the term nanofluid that represents the fluid in which nanoscale particles are suspended in the base fluid with low thermal conductivity such as water, ethylene glycol, and oil. In recent years, many researchers have studied and reported nanofluid technology experimentally or numerically in the presence of heat transfer.

Heat transfer of a nanofluid flow which is squeezed between parallel plates was investigated analytically using Homotopy perturbation method (HPM) by Sheikholeslami and Ganji [2]. They reported that Nusselt number has direct relationship with nanoparticle volume fraction, the squeeze number and Eckert number when two plates are separated but it has reverse relationship with the squeeze number when two plates are squeezed.

Most of engineering problems, especially some heat transfer equations are nonlinear, therefore some of them are solved using numerical solution and some are solved using the different analytic method, such as perturbation method (PM), homotopy perturbation method (HPM), variational iteration method (VIM). Therefore, many different methods have recently introduced some ways to eliminate the small parameter. One of the semi-exact methods which does not need small parameters is the HPM. The homotopy perturbation method proposed and improved by He [3]. The method yields a very rapid convergence of the solution series in the most cases. The HPM proved its capability to solve a large class of nonlinear problems efficiently, accurately and easily with approximations convergency very rapidly to solution. Usually, few iterations lead to high accuracy solution. This method is employed for many researches in engineering sciences. HPM was used for solving many problems such as : nonlinear MHD Jeffery Hamel problem by Moghimi et al. [4]. Mustafa et al. [5] analyzed the effect of squeezing nanofluid flow between parallel plates...

The main objective of this present study is to investigate nanofluid flow and heat transfer between two parallel flat plates without presence of magnetic field. In this study, we have applied Runge-Kutta- Fehlberg fourth-fifth-order method with shooting technique (RKF45) and homotopy perturbation method (HPM), to find the solution of nonlinear differential equations. The effects of governing parameters such as squeeze number, and nanoparticle volume fraction on velocity, and temperature, as well as on skin-friction coefficient, Nusselt are investigated.

Table 1 : Thermophysical properties of pure water and nanoparticles.

	$\rho$ (Kg/m <sup>3</sup> )	$C_p$ (J/kgK)	$k$ (W/mK)
Pure water	997.1	4179	0.613
Copper (Cu)	8933	385	401

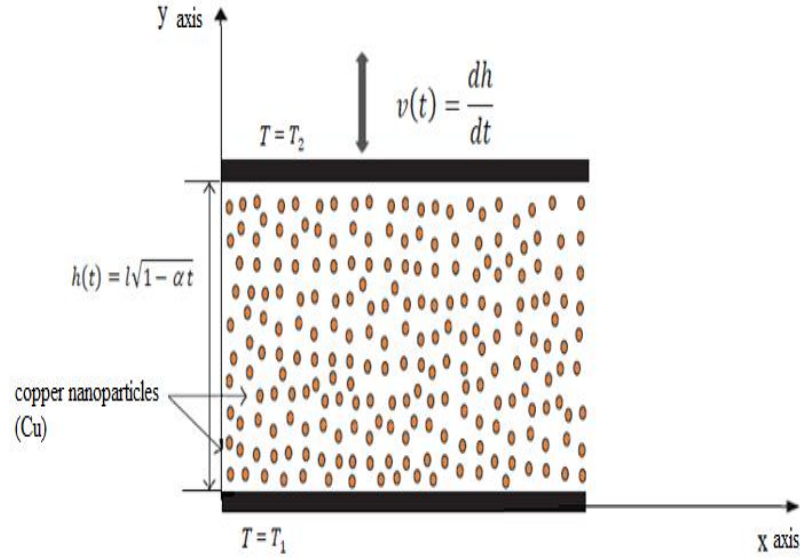


Figure 1: Flow configuration and coordinate system.

## 2. Mathematical Formulation

We consider an unsteady two-dimensional flow to observe heat and mass transfer of a squeezing nanofluid in the middle of two parallel plates extended infinitely and implanted in a system occupied with nanofluid (water as a base fluid) and (copper as a nanoparticle). The thermophysical properties of the nanofluids are given in Table 1. The distance between two plates is  $y = l\sqrt{1 - \alpha t} = h(t)$  where  $l$  is the initial position (at time  $t = 0$ ). Flow is incompressible with no chemical reaction in system. The graphical model support to the present study has been given in Figure 1. The governing equations representing flow are as follows :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\rho_{nf} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial x} + \mu_{nf} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2)$$

$$\rho_{nf} \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial P}{\partial y} + \mu_{nf} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (3)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k_{nf}}{(\rho c_p)_{nf}} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (4)$$

The associated boundary conditions are given as :

$$u = 0, v = 0, T = T_1 \quad \text{en} \quad y = 0 \quad (5)$$

$$u = 0, v = \frac{dh}{dt}, T = T_2 \quad \text{en} \quad y = h(t) \quad (6)$$

Where :

$$\begin{aligned} \rho_{nf} &= (1 - \phi)\rho_f + \phi\rho_p \\ (\rho Cp)_{nf} &= (1 - \phi)(\rho Cp)_f + \phi(\rho Cp)_p \\ \mu_{nf} &= \frac{\mu_f}{(1 - \phi)^{2.5}} \end{aligned} \quad (7)$$

$$\frac{K_{nf}}{K_f} = \frac{K_s + 2K_f - 2\phi(K_f - K_s)}{K_s + 2K_f + 2\phi(K_f - K_s)}$$

Equations (2) – (4) can be converted to a system of nonlinear ordinary differential equations via the following similarity variables :

$$\eta = \frac{y}{l(1 - \alpha t)^{\frac{1}{2}}} = \frac{y}{h(t)} \quad (8)$$

$$u = \frac{\alpha x}{2(1 - \alpha t)} f'(\eta) \quad (9)$$

$$v = \frac{-\alpha l}{2(1 - \alpha t)^{\frac{1}{2}}} f(\eta) \quad (10)$$

$$\theta = \frac{T - T_2}{T_1 - T_2} \quad (11)$$

The transformed equations are :

$$\begin{aligned} f'''' - S(1 - \phi)^{2.5}(3f'' + \eta f'''' + f'f'' - ff''') &= 0 \\ \theta''(\eta) + S Pr \frac{A_2}{A_3}(f\theta' - \eta\theta') &= 0 \end{aligned} \quad (12)$$

where  $A_2$ , and  $A_3$  are dimensionless constants defined as follows :

$$A_2 = \left[ (1 - \phi) + \phi \frac{(\rho C p)_p}{(\rho C p)_f} \right] \quad (13)$$

$$A_3 = \frac{K_{nf}}{K_f}$$

The boundary conditions (5) in the terms of similarity variables (7) become :

$$f'(0) = 0, f(0) = 0, \theta = 1 \text{ at } \eta = 0 \quad (14)$$

$$f'(1) = 0, f(1) = 1, \theta = 1 \text{ at } \eta = 1$$

where  $S = \frac{\alpha l^2}{2\nu_f}$  is the squeeze number,  $Pr = \frac{(\rho C p)_f}{K_f} \nu_{nf}$  is the Prandtl number.

The physical quantities of interest are the skin-friction coefficient  $C_f$ , the Nusselt number  $N_{ux}$ , defined as :

$$C_f = \frac{\tau_w}{\rho_{nf} \nu_w} \quad (15)$$

$$N_{ux} = \frac{l q_w}{k_f (T_1 - T_2)}$$

Where :

$$\tau_w = \mu_{nf} \left( \frac{\partial u}{\partial y} \right)_{y=0} \quad (16)$$

$$q_w = -k_{nf} \left( \frac{\partial T}{\partial y} \right)_{y=0}$$

Using (16) in (15), we get :

$$C_f^* = \frac{x^2}{l^2} (1 - \alpha t) R_{ex} C_f = \frac{f''(0)}{(1 - \phi)^{2.5} A_1} \quad (17)$$

$$N_{ux}^* = \sqrt{1 - \alpha t} N_{ux} = -A_3 \theta'(0)$$

Where  $R_{ex} = \frac{\alpha l^5}{2x^3(1-\alpha t)^2 \nu_f}$  the local Reynolds number.

### 3. Method of Solution

To illustrate the basic ideas of this method, we consider the following equation :

$$A(u) - f(r) = 0, r \in \Omega \quad (18)$$

with the boundary condition of :

$$B \left( u, \frac{\partial U}{\partial n} \right) = 0, \quad r \in \Gamma \quad (19)$$

where  $A$  is a general differential operator,  $B$  a boundary operator,  $f(r)$  a known analytical function and  $\Gamma$  is the boundary of the domain  $\Omega$ .

$A$  can be divided into two parts which are  $L$  and  $N$ , where  $L$  is linear and  $N$  is nonlinear. Eq. (13) can therefore be rewritten as follows :

$$L(u) + N(u) - f(r) = 0, r \in \Omega \quad (20)$$

Homotopy perturbation structure is shown as follows :

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \quad (21)$$

Where :

$$v(r, p): \Omega \times [0, 1] \rightarrow R$$

In Eq. (21),  $p \in [0, 1]$  is an embedding parameter and  $u_0$  is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (15) can be written as a power series in  $p$ , as following :

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots \quad (22)$$

and the best approximation for solution is :

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (23)$$

#### 4. Implementation of the method

According to the so-called homotopy-perturbation method (HPM), we construct a homotopy suppose the solution of Eq. (17) has the form :

$$H(f, p) = (1 - p)[f'''' - f_0''''(0)] + p[f'''' + S A_1(1 - \phi)^{2.5}[3f'' + \eta f''' + f'f'' - ff''']] = 0 \quad (24)$$

$$H(\theta, p) = (1 - p)[\theta'' - \theta_0''(0)] + p \left\{ \theta'' + S Pr \frac{A_2}{A_3} [f\theta' - \eta\theta'] \right\} = 0$$

We consider  $f$  and  $\theta$  as follows :

$$f(\eta) = f_0(\eta) + pf_1(\eta) + \dots = \sum_{i=0}^N f_i(\eta) \quad (25)$$

$$\theta(\eta) = \theta_0(\eta) + p\theta_1(\eta) + \dots = \sum_{i=0}^N \theta_i(\eta)$$

with substituting  $f, \theta$  from equations (20-21) into equations (18-19) and some simplification and rearranging based on powers of  $p$ -terms, we have :

$$p^0 :$$

$$f^{iv} = 0 \quad (26)$$

$$\theta_0'' = 0$$

And boundary conditions are :

$$f(0) = 0, f'(0) = 0, f(1) = 1, f'(1) = 0, \theta(0) = 1 \text{ et } \theta(1) = 0 \quad (27)$$

$P^1 :$

$$f_1^{iv} - 1.102087119S(\eta f_0'''' + 3f_0'' + f_0'f_0'' - f_0f_0''') = 0 \quad (28)$$

$$\theta_1'' + 0.9201899729PrS(\theta_0'f_0 - \eta\theta_0') = 0$$

And boundary conditions are :

$$f(0) = 0, f'(0) = 0, f(1) = 1, f'(1) = 0, \theta(0) = 1 \text{ et } \theta(1) = 0 \quad (29)$$

Solving equations (28) and (26) with boundary conditions, we have :

$$f_0 = -2\eta^3 + 3\eta^2 \quad (30)$$

$$\theta_0 = -\eta + 1$$

$$f_1 = \frac{3306261357}{500000000} S \left( \frac{1}{105} \eta^7 - \frac{1}{30} \eta^6 - \frac{1}{60} \eta^5 + \frac{1}{8} \eta^4 \right) - \frac{472323051}{625000000} \eta^3 S + \frac{157441017}{800000000} \eta^2 S \quad (31)$$

$$\theta_1 = -\frac{9201899729}{10000000000} SPr \left( \frac{1}{10} \eta^5 - \frac{1}{4} \eta^4 + \frac{1}{6} \eta^3 \right) + \frac{9201899729}{60000000000} SPr\eta$$



The terms  $f_i(\eta), \theta_i(\eta)$  when  $i > 2$  are too large that is mentioned graphically.

The solution of this equation, when  $p \rightarrow 1$ , will be as follows :

$$\begin{aligned}
 f(\eta) &= f_0(\eta) + pf_1(\eta) + \dots = \sum_{i=0}^N f_i(\eta) \\
 \theta(\eta) &= \theta_0(\eta) + p\theta_1(\eta) + \dots = \sum_{i=0}^N \theta_i(\eta)
 \end{aligned}
 \tag{32}$$

### 5. Results and discussion

In this paper, Runge-Kutta and HPM Method is applied to solve the problem of the magnetohydrodynamic squeezing flow of nanofluid between parallel plate. The effects of active parameters such as squeeze number, nanoparticle volume fraction are investigated.

Figure. 2 shows the effect of increasing the squeeze number  $S$  on the profile of the velocity, the latter increasing to the movable plate is in the middle of the initial distance between the plates, then it decreases slightly and tends towards zero when the two plates are glued together, and we also observe that the speed decreases with increasing values of  $S$ .

It can easily be seen in Figure 3 that the value of the temperature near the bottom surface of the plate decreases steadily with the increase of the value of  $S$ , and that as we move away from the lower surface of the plate, this value increases.

Figure 4 shows the effect of increasing the volume fraction on the velocity profile, which increases to the moving plate in the middle of the initial distance between the plates, then decreases slightly and tends to zero when the two plates are glued together, and we also observe that the speed decreases with increasing values of  $\phi$ .

Given the previous temperature curve, there is practically no change in the temperature profile when  $\phi$  varies, but there is an effect well shown in the table 2, initially present an increase in the values of the volume fraction of the nanoparticles  $\phi$ , the temperature decreases, and after a fixed distance from the lower surface of the plate, it increases slightly.

From Table 3 it is evident that the coefficient of friction and the Nusselt number are proportional to  $S$ , and it is observed that the parameters  $C_f^*$  increase with increasing values of  $S$  and  $N_u^*$  decrease as  $S$  increases.

In Table 4 it is evident that the coefficient of friction and the Nusselt number are inversely proportional to  $\phi$ ,  $C_f^*$  and  $N_u^*$  decreases as  $\phi$  increases.

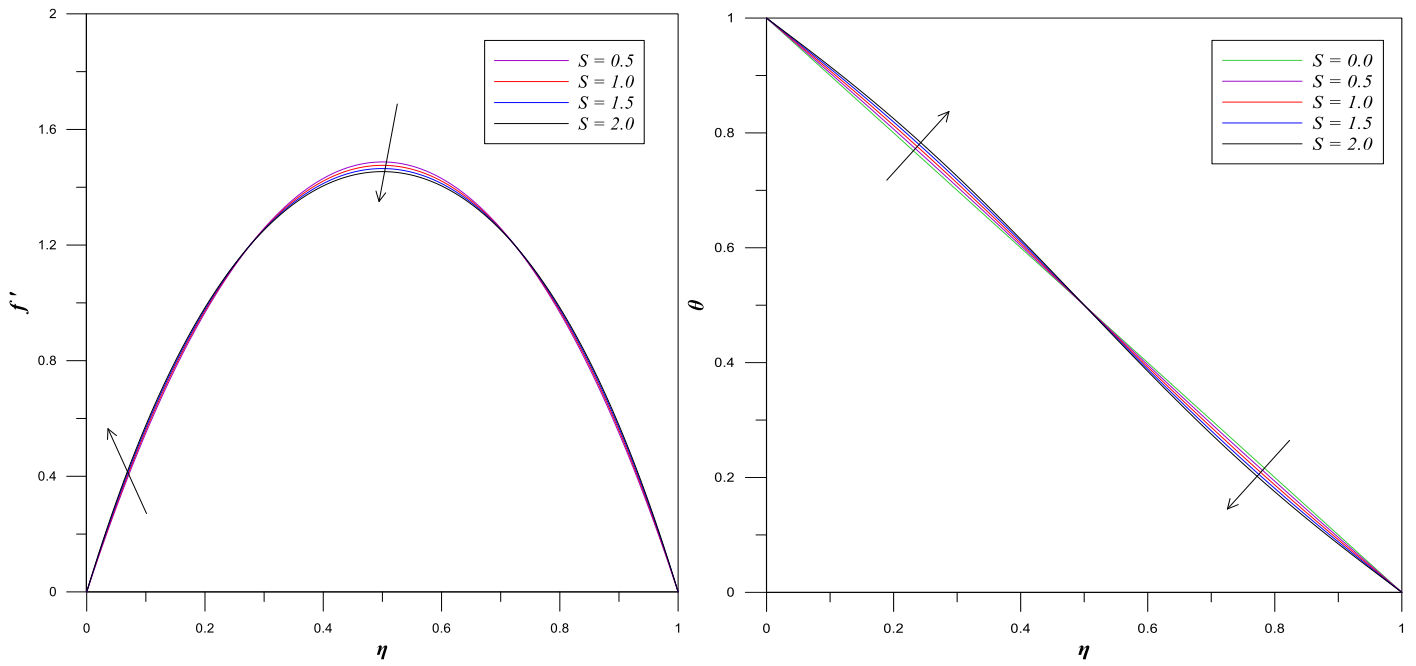


Figure. 3 : velocity and temperature profiles for different squeeze number values ( $S$ ).

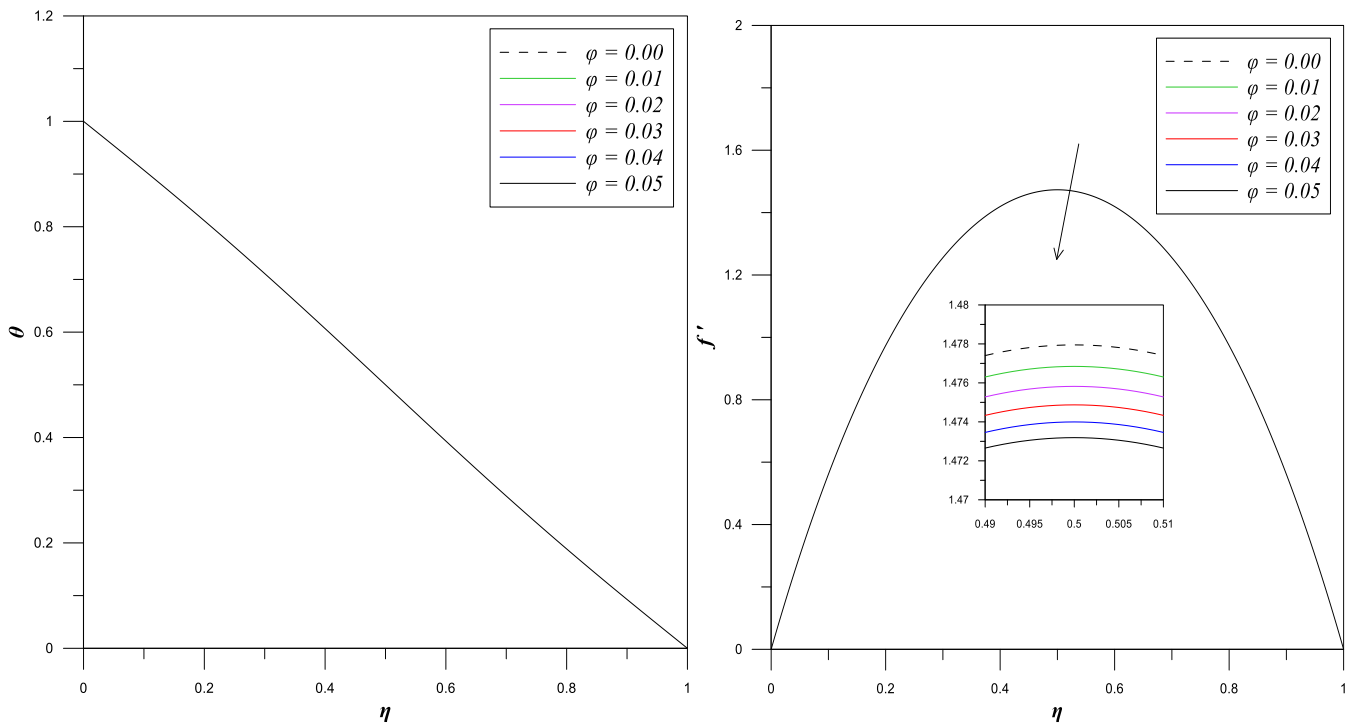


Figure. 4 : velocity and temperature profiles for different volume fraction values ( $\phi$ ).

Table.2 : Temperature value for different values of  $\phi$ .

$\eta$	N = 3, S = 1					
	$\varphi = 0.00$	$\varphi = 0.01$	$\varphi = 0.02$	$\varphi = 0.03$	$\varphi = 0.04$	$\varphi = 0.05$
	$\Theta$	$\Theta$	$\Theta$	$\Theta$	$\Theta$	$\Theta$
0	1	1	1	1	1	1
0.1	0.9089796135	0.9086112279	0.9079192365	0.9082580644	0.9075939228	0.9072813605
0.2	0.815073728	0.8135108681	0.8124090419	0.8129481537	0.8118921319	0.8113961267
0.3	0.7137466214	0.7131637223	0.7120736244	0.7126065942	0.7115633264	0.7110743294
0.4	0.6083624809	0.6080035292	0.6073333522	0.6076608342	0.6070201245	0.6067202682
0.5	0.5000000001	0.5000000001	0.4999999997	0.5000000004	0.4999999996	0.4999999996
0.6	0.3916375189	0.391996471	0.3926666473	0.3923391662	0.3929798746	0.3932797318
0.7	0.2862533788	0.2868362781	0.2879263752	0.2873934062	0.2884366725	0.2889256704
0.8	0.1859012994	0.1864891321	0.187590958	0.1870518467	0.1881078676	0.1886038731
0.9	0.0910203874	0.0913887706	0.187590958	0.0917419333	0.0924060768	0.09271864
1	0	0	0	0	0	0

Table.3: Variation of  $C_f^*$  and  $N_u^*$  for different values of S.

S	N = 4, $\varphi = 0.02$	
	$C_f^*$	$N_u^*$
0.5	5.61973	0.88071
1	5.78950	0.84032
1.5	5.95406	0.80209
2	6.11394	0.76587

Table.3: Variation of  $C_f^*$  and  $N_u^*$  for different values of  $\varphi$ .

$\varphi$	N = 3, S = 1	
	$C_f^*$	$N_u^*$
0	6.34632	0.90222
0.01	6.04486	0.87082
0.02	5.78950	0.84032
0.03	5.57157	0.81069
0.04	5.38450	0.78190
0.05	5.22323	0.75391

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