Interfacial Cracks in Dissimilar Orthotropic Materials

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Abstract: The critical state of composite structures features fracture propagation caused by cracks at the interfaces. The conventional assumption based on the fully open-crack model induces to physically unacceptable behavior that oscillatory singularities in stress and displacement fields lead to interpenetration of both faces. These singularities can be neglected by defining the frictionless contact zones of certain length at the tips of interface cracks. Therefore, an analytical treatment based on TING´s approach is presented for fully open-crack and contact-zone model.

Keywords: Anisotropic elasticity, dislocations, singularities, interfacial cracks.

I. INTRODUCTION

The present paper deals with the two-dimensional problem of fracture behavior at the interface of dissimilar anisotropic materials. It attracts considerable attention in understanding and predicting the macroscopic failure modes evoked by material imperfections, such as cracks, delamination and debonding of adhesive joints in composite materials and sandwich material systems. Analytical investigations for the two-dimensional interface crack problems have been firstly proposed by [1] for isotropic materials, who applied an asymptotic analysis of the elastic fields near the tips of an open interface crack. It has been discovered, that the stress fields and displacements exhibit the so-called oscillatory character of the type $r^{-1/2}$ *sin (or cos)* of the argument $(\lambda_j \cdot \log r)$, where λ_j is a function of material constants and r is the radial distance from the crack tip. Another model which considered partial closed cracks has been proposed by [3], the so-called the contact-zone model, eliminates physically unreal singularities of the classical model.

By virtue of constructive advantages in increasing the stiffness and structural strength and in decreasing the cumulative weight, engineered materials with controlled grain orientation, e.g. periodically layered composites, are nowadays widely applicable in modern light building construction, such as for the aircraft structure. However, the influence of grain direction and the interaction between matrix and fibre complicates the determination of strength and failure modes of anisotropic materials. The propagation of delamination crack describes also a weak and critical link in the reliability and safety of the structure. The above-mentioned methods for isotropic materials are also applicable for anisotropic materials. It has been demonstrated, that an one-dimensional square-root singularity and a two-dimensional oscillatory singularity occur in elastic fields near the tips of an interface cracks [4].

The aim of this paper is to give an analytical review of the interfacial cracks of dissimilar orthotropic bimaterial in consideration of the open-crack model and the contact-zone model based on LEKHNITSKII-ESHELBY-STROH (LES) formalism of anisotropic elasticity theory. This formalism employs a pair of holomorphic functions, namely stress function ϕ and displacement function **u**.

II. STRESS SINGULARITIES OF INTERFACIAL CRACKS

Formulation of GREEN's Function for Interfacial Cracks

Consider a two-dimensional bimaterial that consists of two dissimilar anisotropic elastic half-planes in a fixed polar coordinate system (r,θ). These two half-planes are bonded together along interface $\theta = \theta_0 \pm \pi$, whereas material 1 at $r > 0;\ \theta_0 + \pi > \theta > \theta_0$ lies above material 2 at $r > 0;\ \theta_0 - \pi < \theta < \theta_0$. The boundary conditions of the bimaterial at the interface $\theta = \theta_0 \pm \pi$ subjected to a line force **f** and a line dislocation with BURGERS vector **b** are:

$$\begin{aligned} \mathbf{u}_1(r,\theta_0) &= \mathbf{u}_2(r,\theta_0) \\ \phi_1(r,\theta_0) &= \phi_2(r,\theta_0) \\ \mathbf{u}_1(r,\theta_0+\pi) - \mathbf{u}_2(r,\theta_0-\pi) &= \mathbf{b} \\ \phi_1(r,\theta_0+\pi) - \phi_2(r,\theta_0-\pi) &= \mathbf{f} \end{aligned} \tag{1}$$

where the subscripts 1 and 2 stand for the material 1 and 2, respectively. The general solution of GREEN's function for the infinite space subjected to a line force **f** and a line dislocation with BURGERS vector **b** can be extended for interface crack problem of bimaterial by incorporating the interface orientation θ_0. It appears for material 1 as

$$2\mathbf{u}_1 = -\frac{1}{\pi}(\ln r)\mathbf{h} - \left[\mathbf{S}_1(\theta) - \mathbf{S}_1(\theta_0)\right]\mathbf{h}$$
$$-\left[\mathbf{H}_1(\theta) - \mathbf{H}_1(\theta_0)\right]\mathbf{g} \tag{2}$$
$$2\phi_1 = -\frac{1}{\pi}(\ln r)\mathbf{g} + \left[\mathbf{L}_1(\theta) - \mathbf{L}_1(\theta_0)\right]\mathbf{h}$$
$$-\left[\mathbf{S}_1^T(\theta) - \mathbf{S}_1^T(\theta_0)\right]\mathbf{g}$$

and for material 2 as

$$2\mathbf{u}_2 = -\frac{1}{\pi}(\ln r)\mathbf{h} - \left[\mathbf{S}_2(\theta) - \mathbf{S}_2(\theta_0)\right]\mathbf{h}$$
$$-\left[\mathbf{H}_2(\theta) - \mathbf{H}_2(\theta_0)\right]\mathbf{g} \tag{3}$$
$$2\phi_2 = -\frac{1}{\pi}(\ln r)\mathbf{g} + \left[\mathbf{L}_2(\theta) - \mathbf{L}_2(\theta_0)\right]\mathbf{h}$$
$$-\left[\mathbf{S}_2^T(\theta) - \mathbf{S}_2^T(\theta_0)\right]\mathbf{g}$$

where the vector **h** defines the direction of infinite displacement at point of origin $r = 0$ and the vector **–g** defines the direction of the traction vector $\mathbf{t}_\theta$ on any radial plane $\theta = const.$ Substitution (2) and (3) into (1) yields

$$\begin{aligned} -\left(\mathbf{S}_1 + \mathbf{S}_2\right)\mathbf{h} - \left(\mathbf{H}_1 + \mathbf{H}_2\right)\mathbf{g} = 2\mathbf{b} \\ \left(\mathbf{L}_1 + \mathbf{L}_2\right)\mathbf{h} - \left(\mathbf{S}_1^T + \mathbf{S}_2^T\right)\mathbf{g} = 2\mathbf{f} \end{aligned} \tag{4}$$

where eq. (4) indicates that the vector **h** and **–g** do not depend on the interface orientation θ_0, so that that the prevailed stresses are invariant with the orientation θ_0. Based on the orthogonal and closure relations between complex matrices **S**, **H** and **L**

$$\mathbf{S} = i\left(2\mathbf{A}\mathbf{B}^T - \mathbf{I}\right) \;; \mathbf{H} = 2i\mathbf{A}\mathbf{A}^T \;; \mathbf{L} = -2i\mathbf{B}\mathbf{B}^T \tag{5}$$

it can derived the vector **h** and **g** as a function of line force **f** and line dislocation **b** as

$$\begin{bmatrix} \mathbf{h} \\ \mathbf{g} \end{bmatrix} = 2\begin{bmatrix} \tilde{\mathbf{S}} & \tilde{\mathbf{H}} \\ -\tilde{\mathbf{L}} & \tilde{\mathbf{S}}^{\mathbf{T}} \end{bmatrix}\begin{bmatrix} \mathbf{b} \\ \mathbf{f} \end{bmatrix} \tag{6}$$

The complex matrices **A** and **B** are deduced by comparing the LEKHNITSKII's and STROH's general anisotropic elasticity solutions. They are:

$$\mathbf{A} = \begin{bmatrix} k_1\xi_1(p_1) & k_2\xi_1(p_2) & k_3\eta_1(p_3) \\ k_1 p_1^{-1}\xi_2(p_1) & k_2 p_2^{-1}\xi_2(p_2) & k_3 p_3^{-1}\eta_2(p_3) \\ k_1 p_1^{-1}\xi_4(p_1) & k_2 p_2^{-1}\xi_4(p_2) & k_3 p_3^{-1}\eta_4(p_3) \end{bmatrix} \tag{7}$$

$$\mathbf{B} = \begin{bmatrix} -k_1 p_1 & -k_2 p_2 & -k_3 p_3 \lambda_3 \\ k_1 & k_2 & k_3\lambda_3 \\ -k_1\lambda_1 & -k_2\lambda_2 & -k_3 \end{bmatrix} \tag{8}$$

whereas λ_α is quotient-factor of general solutions of fourth-order elliptic differential equation in anisotropic elasticity defined in an auxiliary complex domain $z = x_1 + px_2$.

$$\begin{aligned} \lambda_\alpha &= -\frac{l_3(p_\alpha)}{l_2(p_\alpha)} = -\frac{l_4(p_\alpha)}{l_3(p_\alpha)}; \quad \alpha = 1,2 \\ l_2(p) &= a'_{55}p^2 - 2a'_{45}p + a'_{44} \\ l_3(p) &= a'_{15}p^3 - \left(a'_{14} + a'_{56}\right)p^2 + \left(a'_{25} + a'_{46}\right)p - a'_{24} \\ l_4(p) &= a'_{11}p^4 - 2a'_{16}p^3 + \left(2a'_{12} + a'_{66}\right)p^2 - 2a'_{26}p + a'_{22} \end{aligned} \tag{9}$$

as well as for $\xi_\alpha(p_\beta);\ \alpha = 1,2,4;\ \beta = 1,2$ and $\eta_\alpha(p_3)$:

$$\begin{aligned} \xi_\alpha(p_\beta) &= p_\beta^2 a'_{\alpha 1} - p_\beta a'_{\alpha 6} + a'_{\alpha 2} + \lambda_\beta\left(p_\beta a'_{\alpha 5} - a'_{\alpha 4}\right) \\ \eta_\alpha(p_3) &= \lambda_3\left(p_3^2 a'_{\alpha 1} - p_3 a'_{\alpha 6} + a'_{\alpha 2}\right) + \left(p_3 a'_{\alpha 5} - a'_{\alpha 4}\right) \end{aligned} \tag{10}$$

The matrix-components of (6) $\tilde{\mathbf{S}}, \tilde{\mathbf{H}}$ and $\tilde{\mathbf{L}}$ base upon the relations of (5) and (7) - (10). They can be expressed as:

$$\begin{aligned} \tilde{\mathbf{S}} &= -\tilde{\mathbf{H}}\left(\mathbf{S}_1^T + \mathbf{S}_2^T\right)\left(\mathbf{H}_1 + \mathbf{H}_2\right)^{-1} = -\left(\mathbf{L}_1 + \mathbf{L}_2\right)^{-1}\left(\mathbf{S}_1^T + \mathbf{S}_2^T\right)\tilde{\mathbf{L}} \\ \tilde{\mathbf{H}} &= \left\{\left(\mathbf{L}_1 + \mathbf{L}_2\right) + \left(\mathbf{S}_1^T + \mathbf{S}_2^T\right)\left(\mathbf{H}_1 + \mathbf{H}_2\right)^{-1}\left(\mathbf{S}_1 + \mathbf{S}_2\right)\right\}^{-1} \\ \tilde{\mathbf{L}} &= \left\{\left(\mathbf{H}_1 + \mathbf{H}_2\right) + \left(\mathbf{S}_1 + \mathbf{S}_2\right)\left(\mathbf{L}_1 + \mathbf{L}_2\right)^{-1}\left(\mathbf{S}_1^T + \mathbf{S}_2^T\right)\right\}^{-1} \end{aligned} \tag{11}$$

Formulation of Stress Singularities for anisotropic material

To define singularities of certain boundaries that contain geometrical discontinuities, such as crack, sharp corners, and jumps in the cross-sectional area in an arbitrary structure, not all analytical solutions provide feasible solutions. However, asymptotic method [1] [5] offers good approximation in solving the boundary value problems regarding prevailed singularities in stress fields. With the aid of the continuously developed finite element method and its supporting information technology, the complicated boundary value problems due to irregular sharp corners can be simply determined.

A general solution of crack as a special corner can be derived by assuming eigenfunctions as homogeneous solutions that satisfy the traction-free boundary conditions at the wedge surfaces. The eigenfunctions for displacement and stress can be assumed as $u \leftrightarrow r^{\delta+1}$ and $t \leftrightarrow r^{\delta}$, respectively. Substituting the complex form of eigenvalue δ

$$\delta = \delta_1 + i\delta_2 \tag{12}$$

in a stress eigenfunction $t \approx r^{\delta}$, it is obvious that for the eigenvalue $\delta_1 < 0$, the stresses oscillate intensely as the distance r approaches the wedge apex $r = 0$. In order to find the eigenfunctions of stress and displacement for anisotropic material, one can apply the STROH's general solutions for the stress function ϕ and displacement function **u**

$$\mathbf{u} = \sum_{\alpha=1}^{3} \left\{ \mathbf{a}_{\alpha} f(z_{\alpha}) \mathbf{q}_{\alpha} + \bar{\mathbf{a}}_{\alpha} \bar{f}(\bar{z}_{\alpha}) \bar{\mathbf{q}}_{\alpha} \right\}$$
$$\phi = \sum_{\alpha=1}^{3} \left\{ \mathbf{b}_{\alpha} f(z_{\alpha}) \mathbf{q}_{\alpha} + \bar{\mathbf{b}}_{\alpha} \bar{f}(\bar{z}_{\alpha}) \bar{\mathbf{q}}_{\alpha} \right\} \tag{13}$$

by assuming function $f(z = \hat{z}) \approx \hat{z}^{\delta+1}$ as singular function in its associated dual complex domains $\hat{z} = \hat{x}_1 + p\hat{x}_2$. Thereafter, for material $i = 1,2$, equation (13) yields to:

$$\mathbf{u}_i = r^{\delta+1} \left\{ \begin{aligned} & \mathbf{A}_i \left(\varsigma^{(i)^{\delta+1}}(\theta,\theta_0) \right) \mathbf{q}_i + \\ & + \bar{\mathbf{A}}_i \left(\bar{\varsigma}^{(i)^{\delta+1}}(\theta,\theta_0) \right) \tilde{\mathbf{q}}_i \end{aligned} \right\}$$
$$\phi_i = r^{\delta+1} \left\{ \begin{aligned} & \mathbf{B}_i \left(\varsigma^{(i)^{\delta+1}}(\theta,\theta_0) \right) \mathbf{q}_i + \\ & + \bar{\mathbf{B}}_i \left(\bar{\varsigma}^{(i)^{\delta+1}}(\theta,\theta_0) \right) \tilde{\mathbf{q}}_i \end{aligned} \right\} \tag{14}$$

where

$$\varsigma^{(i)}(\theta,\theta_0) = \cos(\theta - \theta_0) + p^{(i)}(\theta_0)\sin(\theta - \theta_0) \tag{15}$$

It can be easily proven that the displacement function **u** as well as the stress function ϕ do not depend on the interfacial conditions, whether it is rigidly or flexibly bonded, at $\theta = \theta_0$ and the boundary conditions at $\theta = \theta_0 + \pi$ and $\theta = \theta_0 - \pi$.

$$\varsigma^{(1)^{\delta+1}}(\theta_0 + \pi, \theta_0) = -e^{i\delta\pi}$$
$$\varsigma^{(2)^{\delta+1}}(\theta_0 - \pi, \theta_0) = -e^{-i\delta\pi} \tag{16}$$
$$\varsigma^{(i=1,2)^{\delta+1}}(\theta_0, \theta_0) = 1$$

Thus, the eigenvalue δ is also independent of the orientation of the crack-interface line θ_0. The application of (14) will be described in the next section for a rigidly bonded anisotropic bimaterial. The boundary conditions at interface $\theta = \theta_0$ are:

$$\mathbf{u}_1 = \mathbf{u}_2; \quad \phi_1 = \phi_2 \tag{17}$$

Employing the first equation of (17) to the third equation of (16) and the first equation of (14), one can formulate the vector **h** as:

$$\mathbf{h} = \mathbf{A}_1\mathbf{q}_1 + \bar{\mathbf{A}}_1\tilde{\mathbf{q}}_1 = \mathbf{A}_2\mathbf{q}_2 + \bar{\mathbf{A}}_2\tilde{\mathbf{q}}_2 \tag{18}$$

Similarly, for the vector **g**, it yields as:

$$\mathbf{g} = \mathbf{B}_1\mathbf{q}_1 + \bar{\mathbf{B}}_1\tilde{\mathbf{q}}_1 = \mathbf{B}_2\mathbf{q}_2 + \bar{\mathbf{B}}_2\tilde{\mathbf{q}}_2 \tag{19}$$

The unknown $\mathbf{q}_i$ and $\tilde{\mathbf{q}}_i$ can be deduced from the orthogonal relations between complex matrices **A** and **B**. It appears as:

$$\begin{bmatrix} \mathbf{q}_i \\ \tilde{\mathbf{q}}_i \end{bmatrix} = \begin{bmatrix} \mathbf{B}_i^T & \mathbf{A}_i^T \\ \bar{\mathbf{B}}_i^T & \bar{\mathbf{A}}_i^T \end{bmatrix} \begin{bmatrix} \mathbf{h} \\ \mathbf{g} \end{bmatrix} \tag{20}$$

Substitute (20) in (14) obtains

$$\begin{aligned} \mathbf{u}_i &= r^{\delta+1} \left\{ \mathbf{A}_i \left(\varsigma^{(i)^{\delta+1}}(\theta,\theta_0) \right) \mathbf{B}_i^T + \bar{\mathbf{A}}_i \left(\bar{\varsigma}^{(i)^{\delta+1}}(\theta,\theta_0) \right) \bar{\mathbf{B}}_i^T \right\} \mathbf{h} \\ &+ r^{\delta+1} \left\{ \mathbf{A}_i \left(\varsigma^{(i)^{\delta+1}}(\theta,\theta_0) \right) \mathbf{A}_i^T + \bar{\mathbf{A}}_i \left(\bar{\varsigma}^{(i)^{\delta+1}}(\theta,\theta_0) \right) \bar{\mathbf{A}}_i^T \right\} \mathbf{g} \\ \phi_i &= r^{\delta+1} \left\{ \mathbf{B}_i \left(\varsigma^{(i)^{\delta+1}}(\theta,\theta_0) \right) \mathbf{B}_i^T + \bar{\mathbf{B}}_i \left(\bar{\varsigma}^{(i)^{\delta+1}}(\theta,\theta_0) \right) \bar{\mathbf{B}}_i^T \right\} \mathbf{h} \\ &+ r^{\delta+1} \left\{ \mathbf{B}_i \left(\varsigma^{(i)^{\delta+1}}(\theta,\theta_0) \right) \mathbf{A}_i^T + \bar{\mathbf{B}}_i \left(\bar{\varsigma}^{(i)^{\delta+1}}(\theta,\theta_0) \right) \bar{\mathbf{A}}_i^T \right\} \mathbf{g} \end{aligned} \tag{21}$$

In order to check the properties of the singularities on the right side of point of origin $r = 0$, one can insert $\theta = \theta_0$ in (21) and obtain

$$\mathbf{u}_i(r,\theta_0) = r^{\delta+1}\mathbf{h}; \quad \phi_i(r,\theta_0) = r^{\delta+1}\mathbf{g} \tag{22}$$

Similarly for $\theta = \theta_0 \pm \pi$ preserves the general solution for material 1:

$$\begin{aligned} \mathbf{u}_1(r,\theta_0 + \pi) &= -r^{\delta+1}\left\{ (\cos\delta\pi)\mathbf{h} + (\sin\delta\pi)(\mathbf{H}_1\mathbf{g} + \mathbf{S}_1\mathbf{h}) \right\} \\ \phi_1(r,\theta_0 + \pi) &= -r^{\delta+1}\left\{ (\cos\delta\pi)\mathbf{g} + (\sin\delta\pi)(\mathbf{S}_1^T\mathbf{g} - \mathbf{L}_1\mathbf{h}) \right\} \end{aligned} \tag{23}$$

as well as for material 2:

$$\begin{aligned} \mathbf{u}_2(r,\theta_0 - \pi) &= -r^{\delta+1}\left\{ (\cos\delta\pi)\mathbf{h} - (\sin\delta\pi)(\mathbf{H}_2\mathbf{g} + \mathbf{S}_2\mathbf{h}) \right\} \\ \phi_2(r,\theta_0 - \pi) &= -r^{\delta+1}\left\{ (\cos\delta\pi)\mathbf{g} - (\sin\delta\pi)(\mathbf{S}_2^T\mathbf{g} - \mathbf{L}_2\mathbf{h}) \right\} \end{aligned} \tag{24}$$

The stress singularities incurred in an eigenfunction $f(z) \leftrightarrow \delta$ represent themselves in one of the form of eigenfunctions, when the boundary conditions near the point of origin $r = 0$ of the crack are homogeneous. They may not exist when the boundary conditions at far field disappear. Hence, the character of eigenvalues is peculiar. Therefore, it can be concluded that the presence of singularities depends on the geometry of the crack near the crack apex $r = 0$ and the properties of composite materials.

Next, a case of traction-free crack at the interface will be considered by applying (23) and (24). The boundary condition implies:

$$\phi_1(r,\theta_0+\pi)=\phi_2(r,\theta_0-\pi)=0 \tag{25}$$

and leads to a condition that the vector **g** vanishes. Substitute (23) and (24) in (25) to disregard the influence of vector **g**, one obtains an eigenfunction as a function of vector **h** as:

$$(\sin\delta\pi)\mathbf{h}=0 \tag{26}$$

For non-trivial solution $\delta \neq 0$, it yields to:

$$\left\{\breve{\mathbf{S}}-(\cot\delta\pi)\mathbf{I}\right\}\mathbf{g}=0 \tag{27}$$

where

$$\begin{aligned}&\breve{\mathbf{S}}=\mathbf{D}^{-1}\mathbf{W}\\&\mathbf{D}=\mathbf{L}_1^{-1}+\mathbf{L}_2^{-1}\quad;\quad \mathbf{W}=\mathbf{S}_1\mathbf{L}_1^{-1}-\mathbf{S}_2\mathbf{L}_2^{-1}\end{aligned} \tag{28}$$

Eq. (27) shows that $(\cot\delta\pi)$ is the eigenvalue and **g** the eigenvector of the complex matrix $\breve{\mathbf{S}}$. According to CAYLEY-HAMILTON theory, the characteristic equation for $\breve{\mathbf{S}}$ is:

$$\breve{\mathbf{S}}^3+s^2\breve{\mathbf{S}}=0 \tag{29}$$

where

$$0\le s=\beta=\left\{-\frac{1}{2}tr\left(\breve{\mathbf{S}}^2\right)\right\}^{\frac{1}{2}}<1 \tag{30}$$

Thus, the solution for eigenvalue δ can be obtained as as:

$$\delta=\begin{cases}-\dfrac{1}{2}\\[2mm]-\dfrac{1}{2}\mp i\gamma\end{cases} \tag{31}$$

$$\gamma=\frac{1}{\pi}\tanh^{-1}\beta=\frac{1}{2\pi}\ln\frac{1+\beta}{1-\beta}$$

whereas β is DUNDURS' material parameter.

III. General Oscillatory and Non-Oscillatory Solutions for an Interfacial Crack

General Non-Oscillatory Solution

In this section, a general non-oscillatory solution for an interfacial crack of two-dimensional anisotropic elastic biomaterial will be presented.

Consider an interfacial crack of length $2a$ located at $|x_1|<a;\ x_2=0$. The bimaterial consists of two half-spaces, which half-space 1 $x_2>0$ lies on half-space 2 $x_2<0$. A pair of uniform line traction $\pm t_s$ is employed at the upper and lower surface of the crack on half-space 1 and 2, respectively. The boundary conditions are:

$$\begin{aligned}&\phi_1=\phi_2=0 && at\quad |x|=\infty\\&\phi_1=\phi_2=x_1\cdot t_s && at\quad x_2=\pm 0;\ |x_1|<a\\&u_1=u_2;\ \ \phi_1=\phi_2 && at\quad x_2=0;\ |x_1>a|\end{aligned} \tag{32}$$

By modifying (13), the general solutions of the problem for half-plane $i=1,2$ appears as:

$$\begin{aligned}\mathbf{u}_i&=\mathrm{Re}\left\{\mathbf{A}_i\left(f_o\left(z^{(i)}\right)\right)\mathbf{B}_i^{-1}\right\}\mathbf{t}_s\\\phi_i&=\mathrm{Re}\left\{\mathbf{B}_i\left(f_o\left(z^{(i)}\right)\right)\mathbf{B}_i^{-1}\right\}\mathbf{t}_s\end{aligned} \tag{33}$$

where

$$f_0(z)=\sqrt{z^2-a^2}-z$$

$$f_0(z)=\begin{cases}\pm i\sqrt{a^2-x_1^2}-x_1 & at\quad x_2=\pm 0,\ |x_1|<a\\\pm\sqrt{x_1^2-a^2}-x_1 & at\quad x_2=0,\ \pm x_1>a\end{cases} \tag{34}$$

It is obvious that general solutions (33) satisfy the boundary conditions (32), except the third condition of displacement $u_1=u_2$, where

$$\left(\mathbf{A}_1\mathbf{B}_1^{-1}+\bar{\mathbf{A}}_1\bar{\mathbf{B}}_1^{-1}\right)t_s=\left(\mathbf{A}_2\mathbf{B}_2^{-1}+\bar{\mathbf{A}}_2\bar{\mathbf{B}}_2^{-1}\right)t_s \tag{35}$$

By means of orthogonal relations, the boundary condition (35) may be expressed as:

$$\begin{aligned}&\mathbf{W}\cdot\mathbf{t}_s=0;\\&\mathbf{W}=\mathbf{S}_1\mathbf{L}_1^{-1}-\mathbf{S}_2\mathbf{L}_2^{-1}\end{aligned} \tag{36}$$

The boundary condition (36) is for any line traction $\mathbf{t}_s$ satisfied, when the complex matrices $\mathbf{S}_i$ and $\mathbf{L}_i$ vanish.

However, when the complex matrices $\mathbf{S}_i$ and $\mathbf{L}_i$ do not vanish, namely, $\mathbf{W} \neq 0$, one can define this condition as the mismatching of two homogeneous elastic material. Furthermore, it can be obviously determined that the general non-oscillatory solutions for homogeneous elastic material with crack at the interface of dissimilar bimaterial take place when i) the complex matrices $\mathbf{S}_i$ and $\mathbf{L}_i$ of $i = 1,2$ mismatch, $\mathbf{W} \neq 0$, but the line traction $\mathbf{t}_s$ vanishes, or ii) the complex matrices $\mathbf{S}_i$ and $\mathbf{L}_i$ of both materials do not mismatch. The crack opening displacement $\Delta\mathbf{u}$ can be determined as:

$$\Delta\mathbf{u} = \mathbf{u}(x_1, +0) - \mathbf{u}(x_1, -0) = \sqrt{a^2 - x_1^2}\,\mathbf{D} \cdot \mathbf{t}_s \tag{37}$$

where

$$\mathbf{D} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1} \tag{38}$$

General Oscillatory Solution

The phenomenon of oscillatory displacement indicates a relative small size of interpenetration or overlapping zone between two layers at the interface crack tips. It contradicts the arguments of a real crack of material which has a crack opening displacement between upper and lower layer. The theoretical investigations of open crack problem were firstly performed by [1] using asymptotic analysis of the elastic fields. He found that the stresses and displacements near the tip of the interface cracks contain the degree of singularities (eigenvalue δ) of

$$\delta = -\frac{1}{2} \mp i\gamma \tag{39}$$

By considering (39), the equation (34) will be modified

$$f(z,\gamma) = (z-a)^{\frac{1}{2}+i\gamma}(z+a)^{\frac{1}{2}-i\gamma} - z \tag{40}$$

which leads, depending on the boundary conditions along the x_1-axis, to

$$f(x_1,\gamma) = \begin{cases} \pm\sqrt{x_1^2 - a^2}\, e^{i\gamma \ln\left|\frac{x_1 - a}{x_1 + a}\right|} - x_1 & x_2 = 0; \pm x_1 \rangle a \\ \pm i\sqrt{a^2 - x_1^2}\, e^{\mp\gamma\pi} e^{i\gamma \ln\left|\frac{x_1 - a}{x_1 + a}\right|} - x_1 & x_2 = \pm 0; |x_1| \langle a \end{cases} \tag{41}$$

Thus, the general solutions of the oscillatory singularities for half space 1 in $x_2 > 0$ can be derived as

$$\begin{aligned} \mathbf{u}_1 &= \mathrm{Re}\left\{ e^{\gamma\pi}\mathbf{A}_1\left(f\left(z^{(1)},\gamma\right)\right)\mathbf{B}_1^{-1}\mathbf{d} + e^{-\gamma\pi}\mathbf{A}_1\left(\bar{f}\left(z^{(1)},\gamma\right)\right)\mathbf{B}_1^{-1}\bar{\mathbf{d}} \right\} \\ \phi_1 &= \mathrm{Re}\left\{ e^{\gamma\pi}\mathbf{B}_1\left(f\left(z^{(1)},\gamma\right)\right)\mathbf{B}_1^{-1}\mathbf{d} + e^{-\gamma\pi}\mathbf{B}_1\left(\bar{f}\left(z^{(1)},\gamma\right)\right)\mathbf{B}_1^{-1}\bar{\mathbf{d}} \right\} \end{aligned} \tag{42}$$

and for half space 2 in $x_2 < 0$ as:

$$\begin{aligned} \mathbf{u}_2 &= \mathrm{Re}\left\{ \begin{array}{l} e^{-\gamma\pi}\mathbf{A}_2\left(f\left(z^{(2)},\gamma\right)\right)\mathbf{B}_2^{-1}\mathbf{d} + \\ + e^{\gamma\pi}\mathbf{A}_2\left(\bar{f}\left(z^{(2)},\gamma\right)\right)\mathbf{B}_2^{-1}\bar{\mathbf{d}} \end{array} \right\} \\ \phi_2 &= \mathrm{Re}\left\{ \begin{array}{l} e^{-\gamma\pi}\mathbf{B}_2\left(f\left(z^{(2)},\gamma\right)\right)\mathbf{B}_2^{-1}\mathbf{d} + \\ + e^{\gamma\pi}\mathbf{B}_2\left(\bar{f}\left(z^{(2)},\gamma\right)\right)\mathbf{B}_2^{-1}\bar{\mathbf{d}} \end{array} \right\} \end{aligned} \tag{43}$$

It can be seen that the solutions of (42) and (43) satisfy the boundary conditions (32) if the condition of line traction $\mathbf{t}_s$ appears as

$$t_s = t_0 + t_g \rightarrow t_s = t_g \tag{44}$$

and the vector $\mathbf{d}$ defined as:

$$\mathbf{d} = \frac{\sqrt{1-\beta^2}}{2\beta}\left(\beta\mathbf{t}_g + i\breve{\mathbf{S}}\mathbf{t}_g\right) \tag{45}$$

The general solutions (42) and (43) can also be expressed as functions of complex matrices **S**, **H** and **L.** For stress fields beyond the interface crack at $x_2 = 0; \pm x_1 \rangle a$, one obtains for displacements

$$\begin{aligned} \mathbf{u}_1 &= 2(\cosh\gamma\pi) \\ &\mathrm{Re}\left\{ \left(x_1 \mp e^{i\gamma \ln\left|\frac{x_1 - a}{x_1 + a}\right|}\sqrt{x_1^2 - a^2} \right)\left(\mathbf{S}_1\mathbf{L}_1^{-1} + i\beta\mathbf{L}_1^{-1}\right)\mathbf{d} \right\} \\ \mathbf{u}_2 &= 2(\cosh\gamma\pi) \\ &\mathrm{Re}\left\{ \left(x_1 \mp e^{i\gamma \ln\left|\frac{x_1 - a}{x_1 + a}\right|}\sqrt{x_1^2 - a^2} \right)\left(\mathbf{S}_2\mathbf{L}_2^{-1} - i\beta\mathbf{L}_2^{-1}\right)\mathbf{d} \right\} \end{aligned} \tag{46}$$

and for surface traction vector $(\mathbf{t}_2)_i = \boldsymbol{\sigma}_{i2}$ and the hoop stress vector $(\mathbf{t}_1)_i = \boldsymbol{\sigma}_{i1}$ as:

$$\begin{aligned} &\mathbf{t}_2 = \phi_{,1}; \quad \mathbf{t}_1 = -\phi_{,2} \\ &\mathbf{t}_2 = 2(\cosh\gamma\pi) \\ &\quad \mathrm{Re}\left\{ \left(\pm\eta(x_1)\, e^{i\gamma \ln\left|\frac{x_1 - a}{x_1 + a}\right|} - 1 \right)\mathbf{d} \right\} \end{aligned} \tag{47}$$

$$\mathbf{t}_1^{(1)} = -2(\cosh\gamma\pi)\ \mathrm{Re}\left\{\left(\pm\eta(x_1)e^{i\gamma\ln\left|\frac{x_1-a}{x_1+a}\right|}-1\right)\left(\mathbf{G}_1^{(1)}+i\beta\mathbf{G}_3^{(1)}\right)\mathbf{d}\right\}$$
$$\mathbf{t}_1^{(2)} = -2(\cosh\gamma\pi)\ \mathrm{Re}\left\{\left(\pm\eta(x_1)e^{i\gamma\ln\left|\frac{x_1-a}{x_1+a}\right|}-1\right)\left(\mathbf{G}_1^{(2)}-i\beta\mathbf{G}_3^{(2)}\right)\mathbf{d}\right\} \quad (48)$$

where

$$\cosh\gamma\pi = \frac{1}{\sqrt{1-\beta^2}}: \quad \eta(x_1) = \frac{x_1+2i\gamma a}{\sqrt{x_1^2-a^2}}$$
$$\mathbf{G}_1 = \mathbf{N}_1^{\mathrm{T}} - \mathbf{N}_3\mathbf{S}\mathbf{L}^{-1}; \quad \mathbf{G}_3 = -\mathbf{N}_3\mathbf{L}^{-1} \quad (49)$$
$$\text{where}: \quad \mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T; \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}$$
$$Q_{ik} = C_{i1k1}; \quad R_{ik} = C_{i1k2}; \quad T_{ik} = C_{i2k2}$$

For stress fields in the interface crack at $x_2 = \pm 0; |x_1| \langle a$, the displacements appear as:

$$\mathbf{u}_1 = 2(\cosh\gamma\pi)x_1\,\mathrm{Re}\left\{\left(\mathbf{S}_1\mathbf{L}_1^{-1}+i\beta\mathbf{L}_1^{-1}\right)\mathbf{d}\right\} + 2\sqrt{a^2-x_1^2}\,\mathbf{L}_1^{-1}\,\mathrm{Re}\left\{e^{i\gamma\ln\left|\frac{x_1-a}{x_1+a}\right|}\mathbf{d}\right\}$$
$$\mathbf{u}_2 = 2(\cosh\gamma\pi)x_1\,\mathrm{Re}\left\{\left(\mathbf{S}_2\mathbf{L}_2^{-1}-i\beta\mathbf{L}_2^{-1}\right)\mathbf{d}\right\} - 2\sqrt{a^2-x_1^2}\,\mathbf{L}_2^{-1}\,\mathrm{Re}\left\{e^{i\gamma\ln\left|\frac{x_1-a}{x_1+a}\right|}\mathbf{d}\right\} \quad (50)$$

as well as the traction vectors:

$$\mathbf{t}_2 = -\mathbf{t}_g$$
$$\mathbf{t}_1^{(1)} = 2(\cosh\gamma\pi)\mathrm{Re}\left\{\left(\mathbf{G}_1^{(1)}+i\beta\mathbf{G}_3^{(1)}\right)\mathbf{d}\right\} - \frac{2}{\sqrt{a^2-x_1^2}}\mathbf{G}_3^{(1)}\,\mathrm{Re}\left\{(x_1+2i\gamma a)e^{i\gamma\ln\left|\frac{x_1-a}{x_1+a}\right|}\mathbf{d}\right\} \quad (51)$$
$$\mathbf{t}_1^{(2)} = 2(\cosh\gamma\pi)\mathrm{Re}\left\{\left(\mathbf{G}_1^{(2)}-i\beta\mathbf{G}_3^{(2)}\right)\mathbf{d}\right\} + \frac{2}{\sqrt{a^2-x_1^2}}\mathbf{G}_3^{(2)}\,\mathrm{Re}\left\{(x_1+2i\gamma a)e^{i\gamma\ln\left|\frac{x_1-a}{x_1+a}\right|}\mathbf{d}\right\}$$

The subscripts 1 and 2 and the superscripts (1) and (2) denote respectively the half-spaces 1 and 2. By dint of (28) and the relation of

$$\breve{\mathbf{S}}\mathbf{d} = -i\beta\mathbf{d} \quad (52)$$

it is simple to obtain the crack opening displacement $\mathbf{\Delta u}$ as:

$$\Delta\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2 = 2\sqrt{a^2-x_1^2}\,\mathrm{Re}\left\{e^{i\gamma\ln\left|\frac{x_1-a}{x_1+a}\right|}\mathbf{d}\right\}\mathbf{D} \quad (53)$$

The formulation (53) features that in the interface crack region as the unit point x_1 changes his value in positive axis *0* to *a* as well as in negative axis *0* to *-a*, the crack opening displacement **Δu**, by virtue of logarithmic function, suffers infinite changes in sign repeatedly. The displacement **Δu** oscillates, and this leads to overlapping of the crack faces between two layers.

IV. THE CONTACT-ZONE MODEL

It has been declared in (53) that the oscillations in the interface crack region regarded to the square-root singularity and logarithmic oscillatory singularity experience infinite number of reversal of sign. In order to eliminate the physically undesired oscillatory singularities, COMNINOU [6] proposed the Contact-Zone Model that assumes, unlike the conventional assumption of open interface crack, a partial closed crack. This model assumes that the interface crack of length 2*a* does not fully open under applied loads. At the two ends of respective crack tips, a frictionless contact zone is allowed. It will be further assumed that the normal stress inside the contact tones must be compressive and other related stress singularities vanish. Moreover, the interpenetration of material surfaces may not occur. It can be only achieved, when certain specific boundary conditions are applied in the contact zone, so that the crack interfaces persist in contact at the two ends of respective crack tips.

Considering an interface crack of length 2*a* of a dissimilar bimaterial consists of upper and lower half-spaces as respectively half-space 1 and 2. At infinity in a direction normal to the interface, a uniform tensile stress is applied, so that the cracks open in the interval $(-c,d)$. It is assumed that a frictionless contact of two layers develops in the interval $(-a,-c)$ and (d,a). As preliminary boundary conditions, the normal traction must vanish in interval $(-c,d)$ and the shear traction must vanish in interval $(-a,a)$ [6]. Consequently, a line force **f** will be neglected, yet a single discrete line dislocation with BURGERS vector **b** is applied on the interface $x_1 = 0$. Therefore, equation (6) becomes:

$$\mathbf{h} = 2\tilde{\mathbf{S}}\mathbf{b}; \quad \mathbf{g} = -2\tilde{\mathbf{L}}\mathbf{b} \quad (54)$$

If the line dislocation vector **b** takes place at $r = -r_1$ in half-space 2, it is convenient to replace (3) by substituting (54) as:

$$\phi_2 = \frac{1}{\pi}(\ln r)\tilde{\mathbf{L}}\mathbf{b} + \left\{\mathbf{L}_2(\theta)\tilde{\mathbf{S}} + \mathbf{S}_2^T(\theta)\tilde{\mathbf{L}}\right\}\mathbf{b} \tag{55}$$

where:

$$\begin{aligned} \mathbf{S}(\boldsymbol{\theta}) &= \frac{2}{\pi}\mathrm{Re}\left\{\mathbf{A}\left(\ln\left(\cos\theta + p_* \sin\theta\right)\right)\mathbf{B}^T\right\} \\ \mathbf{L}(\boldsymbol{\theta}) &= -\frac{2}{\pi}\mathrm{Re}\left\{\mathbf{B}\left(\ln\left(\cos\theta + p_* \sin\theta\right)\right)\mathbf{B}^T\right\} \end{aligned} \tag{56}$$

For the boundary conditions along the positive x_1 - axis, it yields:

$$\phi_2(\theta = 0) = \frac{1}{\pi}(\ln r)\tilde{\mathbf{L}}\mathbf{b} \tag{57}$$

and along the negative x_1 - axis:

$$\phi_2(\theta = -\pi) = \frac{1}{\pi}(\ln r)\tilde{\mathbf{L}}\mathbf{b} - \left\{\mathbf{L}\tilde{\mathbf{S}} + \mathbf{S}^T\tilde{\mathbf{L}}\right\}\mathbf{b} \tag{58}$$

Applying (48), one can obtain the traction vector $\mathbf{t}_2$ on the interface $x_2 = 0$ as:

$$\mathbf{t}_2 = \phi_{,1} = \frac{1}{\pi x_1}\tilde{\mathbf{L}}\mathbf{b} + \delta(x_1)\left(\mathbf{L}\tilde{\mathbf{S}} + \mathbf{S}^T\tilde{\mathbf{L}}\right)\mathbf{b} \tag{59}$$

By modifying (59) for a distribution of dislocations with density $b(x_1)$ in the interval $(-a,a)$, one obtains a singular integral equation as:

$$\mathbf{t} = -\frac{1}{\pi}\int_{-a}^{a}\frac{\tilde{\mathbf{L}}\mathbf{b}(\xi)}{\xi - x_1}d\xi + \left(\mathbf{L}\tilde{\mathbf{S}} + \mathbf{S}^T\tilde{\mathbf{L}}\right)\mathbf{b}(x) \tag{60}$$

Assuming that the traction $\mathbf{t}$ is defined as applied stress at infinity $\left(\mathbf{t}^\infty\right)_i = \boldsymbol{\sigma}_{2i}^\infty$, equation (60) yields as:

$$\mathbf{t} = \mathbf{t}^\infty - \frac{1}{\pi}\int_{-a}^{a}\frac{\tilde{\mathbf{L}}\mathbf{b}(\xi)}{\xi - x_1}d\xi + \left(\mathbf{L}\tilde{\mathbf{S}} + \mathbf{S}^T\tilde{\mathbf{L}}\right)\mathbf{b}(x) \tag{61}$$

Thus, for partially open interface crack, the surface traction for interval $(-a,a)$ with following boundary condition

$$b_2(x) = 0 \quad for \quad -a\langle x \langle -c \quad and \quad d \langle x \langle a \tag{62}$$

can be written as:

$$\mathbf{t}_1^\infty = \frac{1}{\pi}\int_{-a}^{a}\frac{\left(\tilde{\mathbf{L}}\mathbf{b}(\xi)\right)_1}{\xi - x}d\xi - \tilde{\mathbf{w}}_3\mathbf{b}_2(x) + \tilde{\mathbf{w}}_2\mathbf{b}_3(x) \tag{63}$$

and for interval $(-c,d)$:

$$\mathbf{t}_2^\infty = \frac{1}{\pi}\int_{-a}^{a}\frac{\left(\tilde{\mathbf{L}}\mathbf{b}(\xi)\right)_2}{\xi - x}d\xi + \tilde{\mathbf{w}}_3\mathbf{b}_1(x) - \tilde{\mathbf{w}}_1\mathbf{b}_3(x) \tag{64}$$

Equation (61) concludes that when normal traction $\mathbf{t}$ in the contact zones is compressive, it verifies the physical correctness COMNINOU´s theory. Moreover, it shows that the overlapping of surface regions of two layers does not occur.

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