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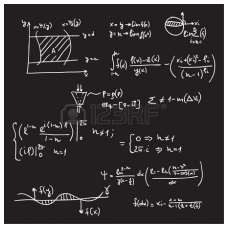

MATHEMATICAL ASSESSMENT OF THE IMPACT OF VACCINATION AND PERSONAL PROTECTION ON THE DYNAMICAL TRANSMISSION OF AVIAN INFLUENZA A (H7N9).

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ABSTRACT: In this article, we investigate the impact of vaccination and personal protection on the dynamical transmission of avian influenza A (H7N9) within a human community. We propose a mathematical model for the dynamical transmission of AI which integrates the key epidemiological and biological features of AI such as vaccine efficacy and the efficiency of personal protection. We provide a theoretical study of the model. We derive the basic reproduction number $\mathcal{R}_0$ which determines the extinction and persistence of the disease. We show that the disease-free equilibrium is globally asymptotically stable whenever $\mathcal{R}_0 \leq 1$, while when $\mathcal{R}_0 > 1$, the disease-free equilibrium is unstable and there exists an endemic equilibrium point which is locally or globally (depending on the case) asymptotically stable on a positively invariant region of the positive orthant. The sensitivity analysis of the model has been performed in order to determine the impact of related parameters on outbreak severity. Theoretical results are supported by numerical simulations, which further demonstrate that some proposed control strategies will not lead to disease eradication, however, if we only employ vaccination, it will require slightly longer to eradicate the disease than applying a combination of pharmaceutical (vaccination) and non-pharmaceutical (personal protection) control methods. In conclusion, it is important to adopt a combination of control methods to fight an avian influenza outbreak.

KEYWORDS: Avian influenza, Mathematical models, Sensitivity analysis, Stability, Personal protection, Vaccination.

1. Introduction

In general, the avian influenza virus does not infect humans. Influenza viruses are widespread and due to their high mutation rate there are many subtypes. In addition, H5N1, H7N4, H7N7, H7N9, H9N2 viruses and other avian influenza viruses with pathogenicity represent a significant potential threat to humans. In particular, the H7N9 subtype virus is mainly transmitted through the respiratory tract. Infected poultry and their secretions, feces and water contaminated with the virus are the main sources of transmission of avian influenza. In February 2013, 3 people were infected for the first time and as of May 31, 132 cases

have been found, including 37 deaths, and the death rate even reaches 30%. [1, 2, 3, 8]. At present, humans infected with avian influenza A (H7N9) are still sporadic and he has yet to find the capacity for the virus to spread among humans. Sporadic infections almost affect poultry mainly in farms, live poultry markets, wet markets and other areas [9, 10, 11, 12, 13]. In humans, the bird flu virus causes symptoms similar to those of other types of flu. These include fever, cough, sore throat, muscle pain, conjunctivitis and, in extreme cases, severe breathing problems and pneumonia which can be fatal [4, 5]. The incubation period of a human infected with the H7N9 influenza virus is about seven days and there are currently drugs available to combat this virus. Although these antiviral drugs are clinically effective against H7N9 avian influenza, mortality from avian influenza H7N9 is still very high. Normally, the H7N9 virus is not thought to have a strong capacity for efficient human to-human propagation, but it were two cases of family aggregation. In such circumstances, it is important to study what may be the best policies available for the prevention and control of transmission avian influenza A (H7N9).

Instead of culling the poultry, control strategies that may control and prevent the spread of avian influenza out to be taken into consideration. Thus, several types of mathematical models have been studied. Nunõ et al [14] analysed a model to examine the role of hospital and community control measures, antiviral drugs and vaccination in combating a potential flu pandemic in a population. Gumel considered the dynamics of a two strain influenza model and conclude that the influenza-related burden in humans increased as the mutation rate increased. Liu et al [15, 16] and Zhang et al [17] modeled the spread of avian influenza H7N9 using both semilinear and half-saturation incidence rate. Chong et al [39] and Liu et al [34] considered saturation incidence rate to investigate the effect on transmission dynamics of avian influenza were both established. Chong et al [39] examined the effect of phamaceutical and non pharmaceutical control strategies whereas Liu et al[34] considered the psychological effect on humans in reponse to the outbreaks of avian influenza (H5N1). Recently, Lee et al [40] modeled the transmission dynamics and control strategies assessment of avian influenza A (H5N6) in the Philippines.

In the present study, motivated by the works of [39, 40], we built an extension of the mathematical model done by [18] by taking into account two control strategies: The personal protection by humans since there are several potential modes of avian influenza transmission such as the consumption of raw or undercooked infected poultry products, contact with oral/nasal mucous membrane or conjunctiva (for example, through swimming or bathing in a contaminated pond/pool), inhalation of contaminated dust or fine water droplets and human-to-human transmission [7]. Although the exact mode of human-to-human transmission remains unclear, there is reason to believe that unprotected contact with an infected person, respiratory secretions, body fluids or waste poses a higher risk for transmission, especially for health-care workers (HCWs) who are first responders [7, 3]. To reduce the mortality and infection rate of avian influenza, the general public especially, health-care workers and workers and employers who are involved in poultry agriculture or have frequent contact with wild birds is advised to follow trict guidelines for personal protection. For example, one should take precautions for hygiene, using gloves, masks and

other protective gear [3]. Controlling and diminishing the spread of avian influenza is a challenging task, as the disease is very infectious and able to mutate into highly pathogenic strains [21]. Consequently, vaccination of poultry or humans as a tool to manage, prevent or eradicate the disease has been recommended by the United Nations [6]. The resulted model is deeply analyzed both theoretically and computationally. From the analytical perspectives, we established the threshold dynamic of the system and transcritical bifurcation using Lyapunov-LaSalle, Poincaré-Bendixson techniques and center manifold approximation, respectively.

The paper is organized as follows. After the formulation of the model in Section 2, we present its quantitative and qualitative analysis in Section 3. Section 4 deals with sensitivity analysis of the model. Section 5 presents a model analysis with personal protection only. Theoretical results are illustrated by numerical simulations in Section 6. The last section is devoted to concluding remarks on how our work fits in the literature and on possible extensions.

2. Model formulation

The dynamics of Avian Influenza (AI) is governed by the following set of biological assumptions: (i) the vaccine confer a total immunity to all vaccine recipients; (ii) vaccinated poultry whose vaccination has failed may be infected with the virus. We consider seven distinct populations, according to their disease status: susceptible poultry S_p (poultry who are susceptible to the disease), vaccinated poultry V_p (healthy poultry who have been vaccinated acquiring immunity), Infectious poultry I_p (infected poultry who show the symptoms of the infection), susceptible humans S_h, latent humans E_h (healthy humans who carry AI virus and are infectious), infectious humans I_h , and the concentration of virus C into the farms environment. Thus, the total poultry and humans population respectively $N_p(t)$ and $N_h(t)$ at time t is

$$N_p(t) = S_p(t) + V_p(t) + I_p(t) \text{ and } N_h(t) = S_h(t) + E_h(t) + I_h(t). \quad (1)$$

Poultry and human are recruited respectively at constant rate Λ_p and Λ_h. A mass vaccination programme may be initiated whenever there is an increase of the risk of an epidemic. The introduction of a vaccine in a poultry population living in an endemic situation is not considered. We suppose that a fraction $0 \leq \pi \leq 1$ of the entire susceptible poultry is being continuously vaccinated. Thus, the population of vaccinated poultry is increased by the vaccination of susceptible poultry at constant rate π.

Most of the theory about disease evolution is based on the assumption that the host population is homogeneous. Poultry hosts, however, may differ and they may constitute very different habitats. In particular, some habitats may provide more resources or be more vulnerable to virus exploitation [56]. The use of models with imperfect vaccines can describe better this type of poultry heterogeneity. The vaccination may reduce but not completely eliminate susceptibility to infection. For this reason, we consider a factor ν as the vaccine efficacy. When $\nu = 1$, the vaccine is perfect while, when $\nu = 0$, the vaccine has no effect at

all. The value $1-\nu$ can be understood as the inefficacy level of the vaccine. Since, a majority of the available vaccines for the human population does not produce 100% success in the disease battle [57, 53, 52, 51], we suppose that the available vaccines for the poultry population does not produce 100% success. Usually, the vaccines are imperfect, which means that a minor percentage of cases, in spite of vaccination, are infected [57, 53]. The susceptible and vaccinated poutry population are decreased due to the AI infection at rates λS_p and $(1-\nu)\lambda V_p$, respectively where λ is the force of infection given by

$$\lambda = \beta_v \frac{I_p}{H_p + I_p} + \beta_e \frac{C}{H_e + C}, \tag{2}$$

β_v is the transmission coefficient, such that $\beta_v I_p$ measures the infection force of the infective poultry, H_p is the half-saturation constant, that is, the density of infected individuals in the population that yields 50% possibility of contracting avian influenza. In the latter saturated incidence function, β_e is the transmission coefficient such that $(\beta_e \gg \beta_v)$; $1/(H_e+C)$ represents saturation due to the cleaning of the farm when the concentration of excretion becomes larger; H_e is the concentration of V. avian viruses attached to aerosol particles in the farm which 50% chance of catching the infection. In fact the transmission potential of the later is higher because they can freely establish contacts with susceptible individuals since they may not be aware of their disease status. The population of infected poultry is increased by the infection of susceptible and vaccinated poultry at rates λS_p and $(1-\nu)\lambda V_p$, respectively, and is diminished by natural death at constant rate δ_p and AI induced poultry mortality at constant rate μ_p. The infected poultry infect the farm at constant rate ϕ and the natural death rate of virus (or shedding rate) is ξ. The susceptible humans are decreased due to the spillover at rates $(1-cq)\tau_p \frac{S_h}{H_{ph} + I_p}$ and $(1-cq)\tau_e \frac{S_h}{H_{eh} + C}$; where τ_p and τ_e is the transmission coefficient of this disease respectively from poultry and pathogenic or infectious environment to humans; Here, $0 \leq c \leq 1$ is the fraction of population that has adopted personal protection and $0 \leq q \leq 1$ is the efficiency of personal protection. For $c = 1$, all the people in a particular community employ personal protection, whereas $c = 0$ means there is no one practicing personal protection. Further, the value $q = 1$ shows that the efficiency of personal protection is 100%. Hence, the values of c and q are reciprocal to the rate of avian influenza transmission [58]. The population of latent humans is increased by the infection of susceptible at rate $(1-cq)\tau_p \frac{S_h}{H_{ph} + I_p}$ and $(1-cq)\tau_e \frac{S_h}{H_{eh} + C}$ and is diminished by natural death at constant rate δ_h and recover (moving to the susceptible class S_h) at rate a. The population of infectious (moving to the infectious class I_h) is increased by latent who develop the disease at rate ϵ and is diminished by recovery from the disease (moving to the susceptible class S_h) at constant rates γ, natural death and spillover induced humans mortality at constant rates δ_h and μ_h, respectively.

A schematic model flowchart is depicted in Figure 1.

From this, the AI transmission model is described by the following system of nonlinear

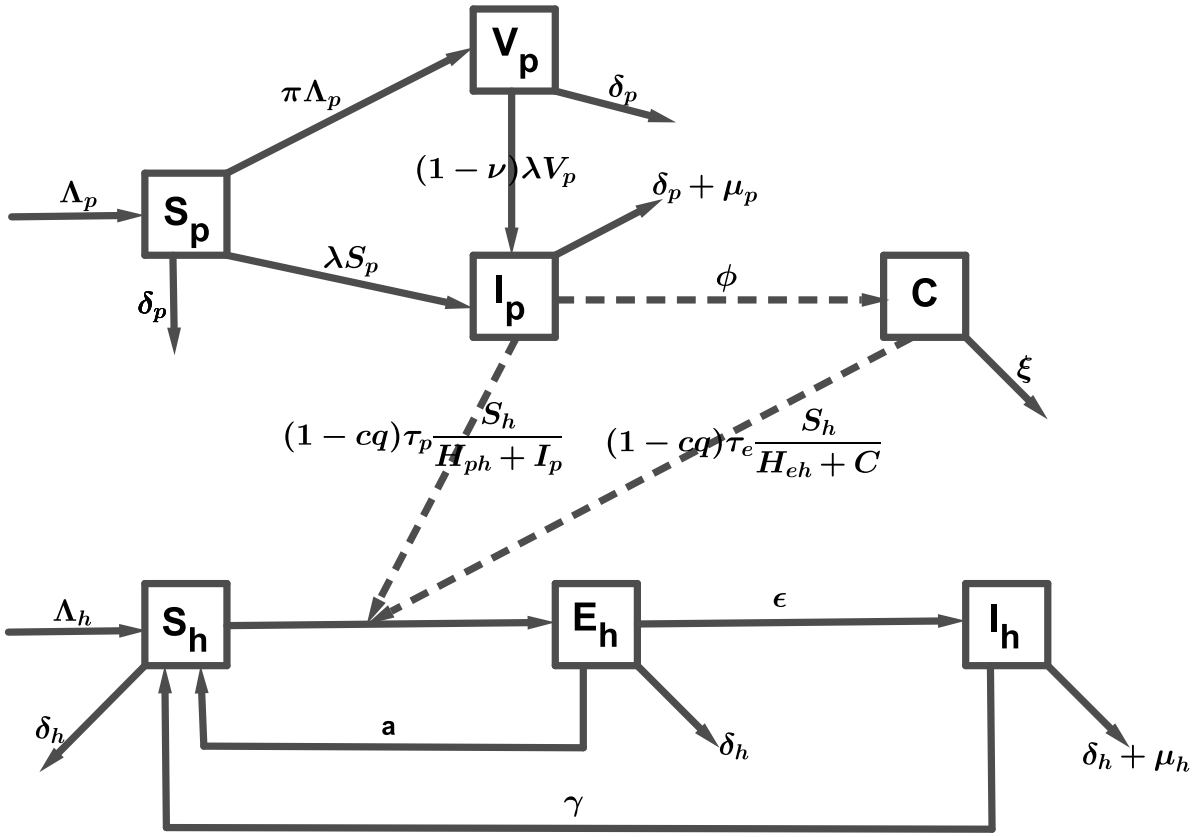

Figure 1: Structure of the model: $S_p - V_p - I_p - S_h - E_h - I_h - C$; $\lambda = \beta_v \dfrac{I_p}{H_p + I_p} + \beta_e \dfrac{C}{H_e + C}$.

ordinary differential equations

$$
\begin{cases}
\dfrac{dS_p}{dt} = (1-\pi)\Lambda_p - (\delta_p + \lambda)S_p, \\
\dfrac{dV_p}{dt} = \pi\Lambda_p - \left[\delta_p + (1-\nu)\lambda\right] V_p, \\
\dfrac{dI_p}{dt} = \left[S_p + (1-\nu)V_p\right]\lambda - (\delta_p + \mu_p)I_p, \\
\dfrac{dS_h}{dt} = \Lambda_h + aE_h + \gamma I_h - (1-cq)\tau_p \dfrac{S_h I_p}{H_{ph} + I_p} - (1-cq)\tau_e \dfrac{S_h C}{H_{eh} + C} - \delta_h S_h, \\
\dfrac{dE_h}{dt} = (1-cq)\tau_p \dfrac{S_h I_p}{H_{ph} + I_p} + (1-cq)\tau_e \dfrac{S_h C}{H_{eh} + C} - (a + \delta_h + \epsilon)E_h, \\
\dfrac{dI_h}{dt} = \epsilon E_h - (\gamma + \mu_h + \delta_h)I_h, \\
\dfrac{dC}{dt} = \phi I_p - \xi C.
\end{cases}
\tag{3}
$$

where λ is the infection force defined in Equation (2). Which is the combination of the poultry system (4) and the human system (5).

$$\begin{cases} \dfrac{dS_p}{dt} = (1-\pi)\Lambda_p - (\delta_p + \lambda)S_p, \\ \dfrac{dV_p}{dt} = \pi\Lambda_p - [\delta_p + (1-\nu)\lambda]\, V_p, \\ \dfrac{dI_p}{dt} = [S_p + (1-\nu)V_p]\,\lambda - (\delta_p + \mu_p)I_p, \\ \dfrac{dC}{dt} = \phi I_p - \xi C. \end{cases} \tag{4}$$

$$\begin{cases} \dfrac{dS_h}{dt} = \Lambda_h + aE_h + \gamma I_h - (1-cq)\tau_p \dfrac{S_h I_p}{H_{ph} + I_p} - (1-cq)\tau_e \dfrac{S_h C}{H_{eh} + C} - \delta_h S_h, \\ \dfrac{dE_h}{dt} = (1-cq)\tau_p \dfrac{S_h I_p}{H_{ph} + I_p} + (1-cq)\tau_e \dfrac{S_h C}{H_{eh} + C} - (a + \delta_h + \epsilon)E_h, \\ \dfrac{dI_h}{dt} = \epsilon E_h - (\gamma + \mu_h + \delta_h)I_h. \end{cases} \tag{5}$$

Table 1 summarizes the model variables and parameters.

3. Mathematical analysis

3.1. Basic properties

Herein, we study the basic properties of the solutions of model system (3), which are essential in the proofs of stability results. We have the following result.

Theorem 1 *Model system (3) is a dynamical system on the biologically feasible compact domain:*

$$\Omega = \left\{(S_p, V_p, I_p, S_h, E_h, I_h, C) \in \mathbb{R}_+^7 / S_p + V_p + I_p \leq M_1 \ ; \ S_h + E_h + I_h \leq M_2 \ ; \ C \leq M_3\right\}.$$

$$M_1 = \max\left\{\frac{\Lambda_p}{\delta_p}; N_p(0)\right\}, \quad M_2 = \max\left\{\frac{\Lambda_h}{\delta_h}; N_h(0)\right\}, \quad M_3 = \max\left\{\frac{\phi M_1}{\xi}; C(0)\right\}$$

Proof. The proof is provided in two steps.

Step 1: We show that the solution variables $(S_p, V_p, I_p, S_h, E_h, I_h, C)$ of model system (3) corresponding to initial conditions such that $S_p(0) > 0, V_p(0) > 0, I_p(0) > 0, S_h(0) > 0, E_h(0) > 0, I_h(0) > 0$ and $C(0) > 0$ are non-negative. Define

$$t_1 = \sup\{t > 0 / \forall u \in [0; t[\ S_p(u) > 0, V_p(u) > 0, I_p(u) > 0, S_h(u) > 0, E_h(u) > 0, I_h(u) > 0, C(0) > 0\}.$$

The initial conditions above and the continuity of the functions $S_p, V_p, I_p, S_h, E_h, I_h, C$ ensure the existence of t_1. If $t_1 = +\infty$ then, all solutions of model system (3) are positive. Suppose $t_1 < \infty$ (t_1 finite), then there is at least one solution component $S_p, V_p, I_p, S_h, E_h, I_h, C$ which is equal to zero at value t_1 (from the definition of t_1 as a supremum).
Suppose for example that $S_h(t_1) = 0$ and let consider the fourth equation of model system (3):

$$\frac{dS_h}{dt} = \Lambda_h + aE_h + \gamma I_h - (1-cq)\tau_p \frac{S_h I_p}{H_{ph} + I_p} - (1-cq)\tau_e \frac{S_h C}{H_{eh} + C} - \delta_h S_h.$$

$$\text{Let } \lambda_1(t) = (1-cq)\tau_p \frac{I_p}{H_{ph} + I_p} + (1-cq)\tau_e \frac{C}{H_{eh} + C}.$$

We know that for all $t \in [0, t_1], \Lambda_h + aE_h(t) + \gamma I_h(t) \geq 0$. It follows that

$$\frac{dS_h}{dt} + (\lambda_1(t) + \delta_h) S_h \geq 0.$$

Therefore

$$\begin{aligned}
\frac{d}{dt}\left[S_h(t)\exp\left\{\delta_h t + \int_0^t \lambda_1(t)ds\right\}\right] &= \dot{S}_h(t)\exp\left\{\delta_h t + \int_0^t \lambda_1(s)ds\right\} \\
&\quad + S_h(t)(\lambda_1(t) + \delta_h)\exp\left\{\delta_h t + \int_0^t \lambda_1(s)ds\right\} \\
&= \exp\left\{\delta_h t + \int_0^t \lambda_1(s)ds\right\} \times \left(\dot{S}_h(t) + (\delta_h + \lambda_1(t))S_h(t)\right) \geq 0,
\end{aligned}$$

that is,

$$\frac{d}{dt}\left[S_h(t)\exp\left\{\delta_h t + \int_0^t \lambda_1(s)ds\right\}\right] \geq 0.$$

Integrating the above inequality from 0 to t_1 gives

$$\int_0^{t_1} \frac{d}{dt}\left[S_h(t)\exp\left\{\delta_h t + \int_0^t \lambda(s)ds\right\}\right] dt \geq 0,$$

or equivalently

$$S_h(t_1)\exp\left\{\delta_h t_1 + \int_0^{t_1} \lambda_1(s)ds\right\} - S_h(0) \geq 0.$$

This yields

$$S_h(t_1) \geq S_h(0)\exp\left\{-\delta_h t_1 - \int_0^{t_1} \lambda_1(s)ds\right\} > 0,$$

which is in contradiction with $S(t_1) = 0$.
The other cases $V_p(t_1) = 0,\ I_p(t_1) = 0,\ S_h(t_1) = 0,\ E_h(t_1) = 0,\ I_h(t_1) = 0$ and $C(t_1) = 0$, lead to the same contradiction.

Hence, $S_p(t) > 0,\ V_p(t) > 0,\ I_p(t) > 0,\ S_h(t) > 0,\ E_h(t) > 0,\ I_h(t) > 0$ and $C(t) > 0\ \ \forall t > 0$.

Step 2: We prove that the total population of poultry and humans at time t, $N_p(t)$ and $N_h(t)$ satisfies the boundedness property $0 < N_p(t) \leq M_1, 0 < N_h(t) \leq M_2$; we also prove that the concentration of virus satisfies the boundedness property $0 \leq C(t) \leq M_3$. We point out that this bound represents the unique equilibrium of the dynamics of the total population in the ideal situation where there is no ongoing infection. By adding the equations of model system (3), one obtains the conservation laws:

$$\begin{cases} \dfrac{dN_p}{dt} = \Lambda_p - \delta_p N_p - \mu_p I_p \leq \Lambda_p - \delta_p N_p, \\ \dfrac{dN_h}{dt} = \Lambda_h - \delta_h N_h - \mu_h I_h \leq \Lambda_h - \delta_h N_h. \end{cases}$$

The application of the Gronwall inequality yields

$$\begin{cases} N_p(t) \leq \dfrac{\Lambda_p}{\delta_p} + \left(N_p(0) - \dfrac{\Lambda_p}{\delta_p} \right) e^{-\delta_p t}, \\ N_h(t) \leq \dfrac{\Lambda_h}{\delta_h} + \left(N_h(0) - \dfrac{\Lambda_h}{\delta_h} \right) e^{-\delta_h t}, \quad \forall t \geq 0. \end{cases} \tag{6}$$

Knowing from (6) that I_p is bounded, we have

$$\frac{dC}{dt} = \phi I_p - \xi C \Rightarrow \frac{dC}{dt} \leq \phi M_1 - \xi C.$$

Once more, application of Gronwall inequality gives

$$C(t) \leq \frac{\phi M_1}{\xi} + \left(C(0) - \frac{\phi M_1}{\xi} \right) e^{-\xi t}, \quad \forall t \geq 0.$$

By comparaison principle, we have the result.

Combining **Step 1** and **Step 2**, Theorem 1 follows from the classical theory of dynamical systems. This concludes the proof. ■ Theorem 1 ensures that the model is well posed since its state variables are positive and the size of the total population does not growth exponentially and is bounded by a value which represents the size of the total population in the ideal situation where there is no infection within the community.

3.2. The DFE and its stability

The DFE for an epidemiological model is an equilibrium such that the disease is absent in the community. Thus, if $Q^0 = (S_p^0, V_p^0, I_p^0, S_h^0, E_h^0, I_h^0, C^0)$ is the DFE of model system (3), then $I_p^0 = E_h^0 = I_h^0 = C^0 = 0$. As a consequence of model system (3), S_p^0, V_p^0 and S_h^0 being solutions of the equations:

$$(1-\pi)\Lambda_p - \delta_p S_p^0 = 0, \quad \pi\Lambda_p - \delta_p V_p^0 = 0, \quad \Lambda_h - \delta_h S_h^0 = 0. \tag{7}$$

which has the unique solution:

$$S_p^0 = \frac{(1-\pi)\Lambda_p}{\delta_p}, \quad V_p^0 = \frac{\pi\Lambda_p}{\delta_p}, \quad S_h^0 = \frac{\Lambda_h}{\delta_h}. \tag{8}$$

In order to investigate the stability properties of the DFE Q^0, we need to compute the reproduction/threshold number $\mathcal{R}_0$ of model system (3). To this end, we apply the method in Van den Driessche and Watmough [36], with (I_p, E_h, I_h, C) and (S_p, V_p, S_h) being the infected and uninfected classes, respectively. We point out that the noninfected classes are the classes of individuals who cannot carry the virus in their body, while the infected classes are the classes of individuals who carry the virus in their body. Using the notations in [36], the matrices $\mathcal{F}$ and $\mathcal{V}$, for the new infection and the remaining transfer are respectively, given by

$$\mathcal{F} = \begin{pmatrix} [S_p + (1-\nu)V_p]\,\lambda \\ (1-cq)S_p\left[\dfrac{\tau_p I_p}{H_{ph}+I_p} + \dfrac{\tau_e C}{H_{eh}+C}\right] \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{V} = \begin{pmatrix} (\delta_p+\mu_p)I_p \\ (a+\delta_h+\epsilon)E_h \\ -\epsilon E_h + (\gamma+\mu_h+\delta_h)I_h \\ -\phi I_p + \xi C \end{pmatrix}.$$

The Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ at the DFE are respectively:

$$D\mathcal{F}(Q^0) = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D\mathcal{V}(Q^0) = \begin{bmatrix} V & 0 \\ V_1 & V_2 \end{bmatrix}.$$

Where

$$F = \begin{bmatrix} \dfrac{\beta_v}{H_p}\left[S_p^0 + (1-\nu)V_p^0\right] & 0 & 0 & \dfrac{\beta_e}{H_e}\left[S_p^0 + (1-\nu)V_p^0\right] \\ \dfrac{(1-cq)\Lambda_h\tau_p}{H_{ph}\delta_h} & 0 & 0 & \dfrac{(1-cq)\Lambda_h\tau_e}{H_{eh}\delta_h} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} \delta_p+\mu_p & 0 & 0 & 0 \\ 0 & (a+\delta_h+\epsilon) & 0 & 0 \\ 0 & -\epsilon & (\gamma+\mu_h+\delta_h) & 0 \\ -\phi & 0 & 0 & \xi \end{bmatrix}.$$

Then, the reproduction number $\mathcal{R}_0$ of model system (3) is the spectral radius of the next generation matrix FV^{-1}, that is

$$\mathcal{R}_0 = \rho(FV^{-1}) = \frac{(\beta_v H_e\xi + \beta_e H_p\phi)\left[S_p^0 + (1-\nu)V_p^0\right]}{H_e H_p \xi(\delta_p+\mu_p)}. \tag{9}$$

Theorem 2 *The DFE Q^0 is locally asymptotically stable in Ω if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof. The eigenvalues of a Jacobian matrix of the vector field described by (3) at Q^0, are the roots of the caracteristic equations

$$(\lambda+\delta_h)(\lambda+(a+\delta_h+\epsilon))(\lambda+(\gamma+\mu_h+\delta_h))=0, \tag{10a}$$

$$\lambda^2+2\lambda\delta_p+\delta_p^2=0, \tag{10b}$$

$$\lambda^2+\lambda\left[\xi+\delta_p+\mu_p-\frac{\beta_v}{H_p}\left[S_p^0+(1-\nu)V_p^0\right]\right]+\xi(\delta_p+\mu_p)-\left[\frac{\beta_v\xi}{H_p}+\frac{\beta_e\phi}{H_e}\right]\left[S_p^0+(1-\nu)V_p^0\right]=0. \tag{10c}$$

The roots $\lambda_1,\lambda_2,\lambda_3$ and λ_4 of the quadratic equation (10b) and (10c) respectively satisfies:

$$\begin{aligned}
\lambda_{1,2} &= -\delta_p,\\
\lambda_3+\lambda_4 &= -\xi-(\delta_p+\mu_p)+\frac{\beta_v}{H_p}\left[S_p^0+(1-\nu)V_p^0\right],\\
&= (\delta_p+\mu_p)(\mathcal{R}_0-1)-\xi-\frac{\beta_e\phi\left[S_p^0+(1-\nu)V_p^0\right]}{H_e\xi(\delta_p+\mu_p)},\\
\lambda_3\times\lambda_4 &= \xi(\delta_p+\mu_p)-\left[\frac{\beta_v\xi}{H_p}+\frac{\beta_e\phi}{H_e}\right]\left[S_p^0+(1-\nu)V_p^0\right]=\xi(\delta_p+\mu_p)(1-\mathcal{R}_0).
\end{aligned}$$

Hence, all the roots of $(10a),(10b)$ and $(10c)$ have negative real part whenever $\mathcal{R}_0<1$. Thus The DFE Q^0 of system (3) is locally asymptotically stable in Ω when $\mathcal{R}_0<1$, but unstable when $\mathcal{R}_0>1$. ■

The biological implication of Theorem 2 is that, a sufficiently small flow of infected individuals will not generate an outbreak of the disease unless $\mathcal{R}_0>1$. For a better control of the disease, the global asymptotic stability (GAS) of the DFE is needed. We use a result of Kamgang and Sallet [59] for the global stability of the DFE for a class of epidemiological models. Following Kamgang and Sallet [59], we write model system (3) in the following form:

$$\begin{cases}\dfrac{dx_1}{dt}=A_1(x)(x_1-x_1^0)+A_{12}(x)x_2,\\ \dfrac{dx_2}{dt}=A_2(x)x_2,\end{cases} \tag{11}$$

where $x_1\in\mathbb{R}_+^3$ is the vector whose components are the number of poultry and humans susceptible individuals including vaccinated poultry and $x_2\in\mathbb{R}_+^4$ denoting (its components) the number of poultry and humans infected individuals including latent, infectious individuals and concentration of aerosol. $x=(x_1,x_2)^T, x_1^0=(S_p^0,V_p^0,S_h^0)$ is the nonzero component of the DFE.

$$A_1(x)=\begin{pmatrix}-\delta_p & 0 & 0\\ 0 & -\delta_p & 0\\ 0 & 0 & -\delta_h\end{pmatrix},\ A_{12}(x)=\begin{bmatrix}-\dfrac{\beta_vH_p}{(H_p+I_p)^2}S_p^0 & 0 & 0 & -\dfrac{\beta_eH_e}{(H_e+C)^2}S_p^0\\ -\dfrac{(1-\nu)\beta_vH_p}{(H_p+I_p)^2}V_p^0 & 0 & 0 & -\dfrac{(1-\nu)\beta_eH_e}{(H_e+C)^2}V_p^0\\ -\dfrac{(1-cq)\tau_pH_{ph}}{(H_{ph}+I_p)^2}S_h^0 & a & \gamma & -\dfrac{(1-cq)\tau_eH_{eh}}{(H_{eh}+C)^2}S_h^0\end{bmatrix}$$

$$A_2(x) = \begin{bmatrix} \dfrac{\beta_v H_p\,[S_p + (1-\nu)V_p]}{(H_p + I_p)^2} - (\delta_p + \mu_p) & 0 & 0 & \dfrac{\beta_e H_e\,[S_p + (1-\nu)V_p]}{(H_e + C)^2} \\ \dfrac{(1-cq)\tau_p H_{ph}}{(H_{ph} + I_p)^2} S_h & -(a + \epsilon + \delta_h) & 0 & \dfrac{(1-cq)\tau_e H_{eh}}{(H_{eh} + C)^2} S_h \\ 0 & \epsilon & -(\gamma + \mu_h + \delta_h) & 0 \\ \phi & 0 & 0 & -\xi \end{bmatrix}.$$

If model system (3) satisfies the conditions $H_1 - H_5$ in [59], then the following result holds.

Theorem 3 *The fixed point $Q^0 = (x_1^0, 0)$ is a globally asymptotically stable equilibrium of model system (3) provided that $\mathcal{R}_0 \leq 1$ and the conditions $H_1 - H_5$ in [59] are satisfied.*

Proof. The result of the Kamgang-Sallet approach [59] uses the algebraic structure of model system (11), namely the fact that $A_1(x)$ and $A_2(x)$ are Metzler matrices. Since in the said approach, the matrix $A_2(x)$ is required to be irreducible, we further restrict the domain of the system to

$$\mathbb{D} = \{(x_1, x_2) \in \Omega, x_1 \neq 0\}. \tag{12}$$

The set $\mathbb{D}$ is positively invariant because only the initial point of any trajectory can have $x_1 = 0$ (see Theorem 1). Indeed, from the first, second and fourth equations of model system (3), one has $S_p^{'} > 0, V_p^{'}$ and $S_h^{'} > 0$ whenever $S_p = 0, V_p = 0$ and $S_h = 0$, respectively. Thus,

$$A_2(x) \text{ is Metzler and irreducible for all x} \in \mathbb{D}. \tag{13}$$

The sub-system:

$$\frac{dx_1}{dt} = A_1(x_1, 0)(x_1 - x_1^0),$$

can be expressed as

$$\begin{aligned} \frac{dS_p}{dt} &= (1-\pi)\Lambda_p - \delta_p S_p, \\ \frac{dV_p}{dt} &= \pi\Lambda_p - \delta_p V_p, \\ \frac{dS_h}{dt} &= \Lambda_h - \delta_h S_h. \end{aligned} \tag{14}$$

Resolving the above equations (14) yields

$$S_p(t) = \frac{(1-\pi)\Lambda_p}{\delta_p} + \left\{ S_p^0 - \frac{(1-\pi)\Lambda_p}{\delta_p} \right\} e^{-\delta_p t}, \quad V_p(t) = \frac{\pi\Lambda_p}{\delta_p} + \left\{ V_p^0 - \frac{\pi\Lambda_p}{\delta_p} \right\} e^{-\delta_p t}, \tag{15a}$$

$$S_h(t) = \frac{\Lambda_h}{\delta_h} + \left\{ S_h^0 - \frac{\Lambda_h}{\delta_h} \right\} e^{-\delta_h t}. \tag{15b}$$

Taking the limit of Equations, (15a) and (15b) when $t \to +\infty$ yields

$$\lim_{t \to +\infty} S_h(t) = \frac{\Lambda_h}{\delta_h}, \ \lim_{t \to +\infty} S_p(t) = \frac{(1-\pi)\Lambda_p}{\delta_p} \text{ and } \lim_{t \to +\infty} V_p(t) = \frac{\pi\Lambda_p}{\delta_p}. \tag{16}$$

Therefore, $x_1^0 = (S_p^0, V_p^0, S_h^0)$ is a GAS equilibrium of the reduced system (14) on the subdomain $\{x \in \mathbb{D}, x_2 = 0\}$. Then, the hypothesis H_2 is satisfied.

The result of Kamgang and Sallet (see [59, Theorem 4.3]) gives the GAS of the DFE of a dissipative system of the form (11) which satisfies (13) and (16) provided there exists a matrix $A_2(x)$ with the following additional properties:

$$\begin{cases} A_2(x) \leq \overline{A}_2, \quad x \in \mathbb{D}, \\ \text{if } A_2(\overline{x}) = \overline{A}_2, \text{ for some } x = (\overline{x}_1, \overline{x}_2)^T \in \mathbb{D} \text{ then } \overline{x}_2 = 0, \\ \alpha(\overline{A}_2) \leq 0. \end{cases} \tag{17}$$

Using the fact that $\frac{S_p}{(H_p+I_p)^2} \leq \frac{S_p^0}{H_p^2}, \frac{V_p}{(H_p+I_p)^2} \leq \frac{V_p^0}{H_p^2}, \frac{S_p}{(H_e+C)^2} \leq \frac{S_p^0}{H_e^2}, \frac{V_p}{(H_e+C)^2} \leq \frac{S_p^0}{H_e^2}$ and $\frac{S_h}{(H_{ph}+I_p)^2} \leq \frac{S_p^0}{H_{ph}^2}$, one has

$$\overline{A}_2(x) = \begin{bmatrix} \dfrac{\beta_v \left[S_p^0 + (1-\nu)V_p^0\right]}{H_p} - (\delta_p + \mu_p) & 0 & 0 & \dfrac{\beta_e \left[S_p^0 + (1-\nu)V_p^0\right]}{H_e} \\ \dfrac{(1-cq)\tau_p}{H_{ph}} S_h^0 & -(a+\epsilon+\delta_h) & 0 & \dfrac{(1-cq)\tau_e}{H_{eh}} S_h^0 \\ 0 & \epsilon & -(\gamma+\mu_h+\delta_h) & 0 \\ \phi & 0 & 0 & -\xi \end{bmatrix}.$$

The equality $A_2(x) = \overline{A}_2$ is possible only when $I_p = C = 0$, which implies that $x_2 = 0$. Therefore, the first and second conditions in (17) hold. Note that $\overline{A}_2$ is a Metzler matrix which satisfies the stability condition of Kamgang and Sallet [59].

From the above condition of $\overline{A}_2$, one can observe that there is a maximum which is uniquely realised in $\mathbb{D}$ at Q^0 and this maximum is then the block of the jocobian of model system (11) at Q^0, corresponding to the matrix $A_2(x)$, and the condition H_4 is satisfied.

Now, we check the condition H_5. Note that the condition $\alpha(\overline{A}_2) \leq 0$ implies that $\overline{A}_2$ is a stable Metzler matrix. We show in Appendix A that the condition $\alpha(\overline{A}_2) \leq 0$ is equivalent to $\mathcal{R}_0 \leq 1$.

We can now apply [59, Theorem 4.3] and conclude that the disease-free equilibrium $(x_1^0, 0)$ is GAS in $\mathbb{D}$. From (12), for the points of $\mathbb{D}$ where $x_2 = 0$, the disease-free equilibrium is GAS on Ω. ■

3.3. Endemic equilibrium and its stability

Herein, we compute the endemic equilibrium and study its stability. To this end, we first rewrite poultry system (4) in the following compact form:

$$\begin{cases} \dfrac{dx(t)}{dt} = \Gamma_1 + A_x x(t) - \lambda \sum_{i=1}^{2} D_i \langle e_i | x(t) \rangle, \\ \dfrac{dy(t)}{dt} = A_y y(t) + \lambda \sum_{i=1}^{2} k \langle e_i | x(t) \rangle, \end{cases} \tag{18}$$

where, $x(t) = (S_p(t), V_p(t))^T$, $y(t) = (I_p(t), C(t))^T$, $\Gamma_1 = ((1-\pi)\Lambda_p, \pi\Lambda_p)^T$, $D_1 = (1,0)^T$, $D_2 = (0,1)^T$, $e_1 = (1,0)$, $e_2 = (0, 1-\nu))$, $k = (1,0)^T, \lambda = \langle B|y\rangle$, $B = \left(\frac{\beta_v}{H_p + I_p}, \frac{\beta_e}{H_e + C}\right)$,

$$A_x = \begin{bmatrix} -\delta_p & 0 \\ 0 & -\delta_p \end{bmatrix} \text{ and } A_y = \begin{bmatrix} -(\delta_p + \mu_p) & 0 \\ \phi & -\xi \end{bmatrix}.$$

Let $Q^* = (x^*, y^*)$ be the positive endemic equilibrium of model system (18). This steady state with $y^* > 0$ is obtained by setting the right-hand side of Eq. (18) to zero, giving:

$$\begin{array}{c} \Gamma_1 + A_x x^* - \lambda^* \sum_{i=1}^{2} D_i \langle e_i | x^* \rangle = 0, \\ A_y y^* + \lambda^* \sum_{i=1}^{2} k \langle e_i | x^* \rangle = 0, \end{array} \tag{19}$$

where λ^* is the force of infection at the endemic equilibrium, given by

$$\lambda^* = \langle B^* | y^* \rangle. \tag{20}$$

Multiplying the second equation of (18) by $-A_y^{-1}$ gives

$$y^* = \lambda^* \sum_{i=1}^{2} \langle e_i | x^* \rangle (-A_y^{-1}) k = \lambda^* \left[S_p^* + (1-\nu) V_p^*\right] (-A_y^{-1}) k. \tag{21}$$

A simple calculation gives $\sum_{i=1}^{2} D_i \langle e_i | x^* \rangle = (S_p^*, (1-\nu) V_p^*)^T$. With this in mind, the first Equation of (19) becomes

$$0 = \Gamma_1 + A_x x^* - \lambda^* \sum_{i=1}^{2} D_i \langle e_i | x^* \rangle = \Gamma_1 + A_x x^* - \lambda^* (S_p^*, (1-\nu) V_p^*)^T = \Gamma_1 - (\lambda^* B_x - A_x) x^*, \tag{22}$$

where

$$B_x = \begin{bmatrix} 1 & 0 \\ 0 & 1-\nu \end{bmatrix} \quad \text{and} \quad \lambda^* B_x - A_x = \begin{bmatrix} \lambda^* + \delta_p & 0 \\ 0 & (1-\nu)\lambda^* + \delta_p \end{bmatrix}.$$

Solving Equation (22) gives

$$x^* = (\lambda^* B_x - A_x)^{-1} \Gamma_1, \tag{23}$$

where

$$(\lambda^* B_x - A_x)^{-1} = \frac{1}{\chi(\lambda^*)} \begin{bmatrix} (1-\nu)\lambda^* + \delta_p + \phi_2 & \phi_2 \\ \phi_1 & \lambda^* + \delta_p + \phi_1 \end{bmatrix},$$

and

$$\chi(\lambda^*) = (1-\nu)\lambda^{*2} + (2-\nu)\delta_p \lambda^* + \delta_p^2. \tag{24}$$

We stress that the coefficients of the quadratic polynomial $\chi(\lambda^*)$ are non-negative. As a consequence, $\chi(\lambda^*)$ is positive for any positive value of λ^*.

From Equation (23), one has

$$S_p^* = \frac{(1-\pi)\Lambda_p}{\chi(\lambda^*)}\left[(1-\nu)\lambda^* + \delta_p\right], \quad V_p^* = \frac{\pi\Lambda_p}{\chi(\lambda^*)}\left[\lambda^* + \delta_p\right]. \tag{25}$$

Also, from Equation (21), one gets

$$I_p^* = \frac{\lambda^*}{\delta_p + \mu_p}\left[S_p^* + (1-\nu)V_p^*\right], \quad C^* = \frac{\phi\lambda^*}{\xi(\delta_p + \mu_p)}\left[S_p^* + (1-\nu)V_p^*\right], \tag{26}$$

where

$$S_p^* + (1-\nu)V_p^* = \frac{\Lambda_p}{\chi(\lambda^*)}\left[(1-\nu)\lambda^* + (1-\pi\nu)\delta_p\right]. \tag{27}$$

Now, from the expression of the force of infection at the endemic equilibrium (20), using Equation (21) yields

$$\begin{aligned}
\lambda^* &= \langle B^*|y^*\rangle = \lambda^*\left[S_p^* + (1-\nu)V_p^*\right]\langle B^*|(-A_y^{-1})k\rangle, \\
&= \lambda^*\left[S_p^* + (1-\nu)V_p^*\right]\left[\frac{\beta_v}{(H_p + I_p^*)(\delta_p + \mu_p)} + \frac{\beta_e\phi}{\xi(H_e + C^*)(\delta_p + \mu_p)}\right],
\end{aligned}$$

which gives

$$\frac{S_p^* + (1-\nu)V_p^*}{\xi(\delta_p + \mu_p)}\left[\frac{\beta_v\xi}{(H_p + I_p^*)} + \frac{\beta_e\phi}{(H_e + C^*)}\right] = 1. \tag{28}$$

Then, using the expression of $I_p^*, C^*, S_p^* + (1-\nu)V_p^*$ given in Equations (26)–(27) and

$$\delta_p\left[S_p^0 + (1-\nu)V_p^0\right] = \Lambda_p(1-\pi\nu), \tag{29}$$

one has

$$\Lambda_p\left[(1-\nu)\lambda^* + (1-\pi\nu)\delta_p\right]\left[\beta_v\xi(H_e + C^*) + \beta_e\phi(H_p + I_p^*)\right] = \xi(\delta_p+\mu_p)\chi(\lambda^*)(H_p+I_p^*)(H_e+C^*)$$

$$\begin{aligned}
0 &= \chi(\lambda^*)\phi\xi(\delta_p + \mu_p)I_p^{*2} \\
&\quad + \left[\chi(\lambda^*)(\delta_p + \mu_p)\xi(H_p\phi + H_e\xi) - \Lambda_p\left[\beta_v\phi\xi + \beta_e\phi\xi\right]\left[(1-\nu)\lambda^* + (1-\pi\nu)\delta_p\right]\right]I_p^* \\
&\quad + H_eH_p\xi^2\chi(\lambda^*)(\delta_p + \mu_p) - \Lambda_p\left[H_e\beta_v\xi^2 + H_p\beta_e\phi\xi\right]\left[(1-\nu)\lambda^* + (1-\pi\nu)\delta_p\right], \\
&= \phi\xi\left[(1-\nu)\Lambda_p\right]^2\lambda^{*4} + 2(1-\nu)\Lambda_p^2\phi\xi\delta_p(1-\pi\nu)\lambda^{*3} + \phi\xi\Lambda_p^2\delta_p^2\lambda^{*2} \\
&\quad + \Lambda_p\xi(1-\nu)\chi(\lambda^*)(\delta_p + \mu_p)(H_p\phi + H_e\xi)\lambda^{*2} + \Lambda_p\xi\delta_p\chi(\lambda^*)(\delta_p + \mu_p)(H_p\phi + H_e\xi)\lambda^* \\
&\quad - \left[(1-\nu)\Lambda_p\right]^2\left[\beta_v\phi\xi + \beta_e\phi\xi\right]\lambda^{*3} - \Lambda_p^2(1-\nu)(1-\pi\nu)\delta_p\left[\beta_v\phi\xi + \beta_e\phi\xi\right]\lambda^{*2} \\
&\quad - \Lambda_p^2(1-\nu)\delta_p\left[\beta_v\phi\xi + \beta_e\phi\xi\right]\lambda^{*2} - \Lambda_p^2\delta_p^2(1-\pi\nu)\left[\beta_v\phi\xi + \beta_e\phi\xi\right]\lambda^* \\
&\quad + \chi^2(\lambda^*)(\delta_p + \mu_p)^2\xi^2H_pH_e - \Lambda_p(1-\nu)\chi(\lambda^*)(\delta_p + \mu_p)\left[H_e\beta_v\xi^2 + H_p\beta_e\phi\xi\right]\lambda^* \\
&\quad - \Lambda_p\chi(\lambda^*)(1-\pi\nu)\delta_p(\delta_p + \mu_p)\left[H_e\beta_v\xi^2 + H_p\beta_e\phi\xi\right].
\end{aligned}$$

It can be shown that the nonzero equilibria of model system (3) satisfy the following equation in term λ^*:

$$a_4(\lambda^*)^4 + a_3(\lambda^*)^3 + a_2(\lambda^*)^2 + a_1\lambda^* + a_0 = 0, \tag{30}$$

where,

$$\begin{aligned}
a_4 &= \phi\xi\left[(1-\nu)\Lambda_p\right]^2, \\
a_3 &= 2(1-\nu)\Lambda_p^2\phi\xi\delta_p - \left[(1-\nu)\Lambda_p\right]^2\left[\beta_v\phi\xi + \beta_e\phi\xi\right], \\
a_2 &= \phi\xi\Lambda_p^2\delta_p^2 + \Lambda_p\xi(1-\nu)\chi(\lambda^*)(\delta_p+\mu_p)(H_p\phi + H_e\xi) - \Lambda_p^2(1-\nu)(1-\pi\nu)\delta_p\left[\beta_v\phi\xi + \beta_e\phi\xi\right] \\
&\quad -\Lambda_p^2(1-\nu)\delta_p\left[\beta_v\phi\xi + \beta_e\phi\xi\right] + H_pH_e\xi^2(1-\nu)\chi(\lambda^*)(\delta_p+\mu_p)^2, \\
a_1 &= \Lambda_p\xi\delta_p\chi(\lambda^*)(\delta_p+\mu_p)(H_p\phi + H_e\xi) - \Lambda_p^2\delta_p^2(1-\pi\nu)\left[\beta_v\phi\xi + \beta_e\phi\xi\right] \\
&\quad -\Lambda_p(1-\nu)\chi(\lambda^*)(\delta_p+\mu_p)\left[H_e\beta_v\xi^2 + H_p\beta_e\phi\xi\right] + \chi(\lambda^*)H_pH_e\delta_p(2-\nu)\xi^2(\delta_p+\mu_p)^2, \\
a_0 &= \chi^2(\lambda^*)(\delta_p+\mu_p)^2\xi^2H_pH_e - \Lambda_p\chi(\lambda^*)(1-\pi\nu)\delta_p(\delta_p+\mu_p)\left[H_e\beta_v\xi^2 + H_p\beta_e\phi\xi\right] \\
&= H_pH_e\xi^2\delta_p^2\chi(\lambda^*)(\delta_p+\mu_p)^2\left(1-\mathcal{R}_0\right).
\end{aligned}$$

Thus, positive endemic vector $(S_p^*, V_p^*, I_p^*, C^*)$ are obtained by solving for λ^* from the equation (30) and substituting the result (positive values of λ^*) into the expressions of the variables of model system (3) at the steady state. Clearly, a_0 is positive or negative depending whether $\mathcal{R}_0$ is less than or greater than unity, respectively. Thus, the number of possible real roots of the polynomial (30) depends on the signs of a_4, a_3, a_2, a_1 and a_0. This can be analysed using the Descarte's Rule of Signs on the polynomial $f(\lambda^*) = a_4(\lambda^*)^4 + a_3(\lambda^*)^3 + a_2(\lambda^*)^2 + a_1\lambda^* + a_0$.

We also have in the human system (5)

$$S_h^* = N_h^* - E_h^* - I^*; \quad I_h^* = \frac{\Lambda_h}{\mu_h} - \frac{\delta_h}{\mu_h}N_h^*; \quad E_h^* = \frac{\Lambda_h\eta_2}{\mu_h\epsilon} - \frac{\eta_2\delta_h}{\mu_h\epsilon}N_h^*; \tag{31}$$

where

$$\eta_1 = a + \delta_h + \epsilon\ ,\ \eta_2 = \gamma + \delta_h + \mu_h.$$

N_h^* is given by

$$N_h^* = \frac{\Lambda_h\left[\alpha_1^*(\alpha_2+\alpha_3) + \alpha_2(a+\epsilon+\delta_h)\right]}{\alpha_1^*(1+\alpha_2\delta_h+\alpha_3\delta_h) + \alpha_2\delta_h(a+\epsilon+\delta_h)}, \tag{32}$$

where

$$\alpha_1^* = (1-cq)\left[\frac{\tau_pI_p^*}{H_{ph}+I_p^*} + \frac{\tau_eC^*}{H_{eh}+C^*}\right];\ \alpha_2 = \frac{\eta_2}{\mu_h};\ \alpha_3 = \frac{\eta_2}{\mu_h\epsilon}. \tag{33}$$

Notice that it is not difficult to show that $N_h^* \leq \Lambda_h/\delta_h$. Thus, the existence of a positive endemic vector (S_h^*, E_h^*, I_h^*).

The various possibilities for the roots of Equation (30) and (32) are summarized in the following lemma.

Lemma 1 *Model system* (3) *could have:*

(i) a unique endemic equilibrium if $\mathcal{R}_0 > 1$*,*

(ii) one or more than one endemic equilibrium if $\mathcal{R}_0 > 1$*,*

(iii) more endemic equilibria if $\mathcal{R}_0 < 1$*,*

(iv) no endemic equilibria if $\mathcal{R}_0 < 1$*.*

The proof of case (i) of Lemma 1 is straightforward and evident. Case (iii) of Lemma 1 indicates the possibility of a backward bifurcation in model system (3) (where the locally asymptotically stable DFE co-exists with a locally asymptotically stable endemic equilibrium when $\mathcal{R}_0 < 1$).

Lemma 1 and Theorem 2 establish that $\mathcal{R}_0 = 1$ is a bifurcation parameter. In fact, across $\mathcal{R}_0 = 1$ the disease-free equilibrium, Q^0 changes its stability property from local stability to unstable (see Theorem 2). In the next result, the Centre Manifold Theory as described by [50, Theorem 4.1] is used to investigate the appearance of the transcritical bifurcation at $\mathcal{R}_0 = 1$ where the stable disease-free equilibrium Q^0 becomes unstable when $\mathcal{R}_0$ crosses 1 from below and gives rise to the stable endemic equilibrium Q^*. We have the following Theorem.

Theorem 4 *The ODE system* (3) *has a transcritical forward bifurcation at* $\mathcal{R}_0 = 1$.

The Proof is stated in Appendix B.

Remark 1 *The application of [50, Theorem 4.1] to prove Theorem 4, also establish the local asymptotic stability of* Q^**, but this result applies only for small values of* $\mathcal{R}_0 > 1$.

3.4. Impact of the poultry vaccination

Herein, we study the effect of the vaccination on model system (3). To do so, let us consider the control technique of constant vaccination of susceptible poultry. Suppose that at time $t = 0$, a proportion π of susceptible poultry is vaccinated with an imperfect vaccine. The basic reproduction number of model system (3) without vaccination (i.e. $\pi = 0$) is

$$\mathcal{R}_0^{pp} = \frac{\Lambda_p(\beta_v H_e \xi + \beta_e H_p \phi)}{H_e H_p \delta_p \xi(\delta_p + \mu_p)}. \tag{34}$$

With this mind, one has

$$\mathcal{R}_0 = (1 - \pi\nu)\mathcal{R}_0^{pp}. \tag{35}$$

Observe that $\mathcal{R}_0 \leq \mathcal{R}_0^{pp}$. The constraint $\mathcal{R}_0 \leq 1$ defines implicitly a critical vaccination proportion $\pi > \pi^c$ that must achieved for disease eradication:

$$\pi^c = \frac{1}{\nu}\left[1 - \frac{1}{\mathcal{R}_0^{pp}}\right], \tag{36}$$

Since vaccination entails costs, to choose the smallest coverage that achieves eradication would be the best option. In this way, the entire population does not need to be vaccinated in order to eradicate the disease (this is the herd immunity phenomenon). Vaccinating at the critical level π^c does not instantly lead to disease eradication. Thus, from a public health perspective, π^c acts as a lower bound on what should be achieved, with higher levels of vaccination leading to a more rapid elimination of the disease. However, a critical vaccination portion $\pi > \pi^c$ is necessary but not sufficient. Thus, to better control the infection, the sufficiently for the eradication of the disease within the community. Note

that the constraint $\mathcal{R}_0 \leq 1$ defines also implicitly a critical vaccine efficacy $\nu > \nu^c$ that must be achieved for eradication of the infection:

$$\nu^c = \frac{1}{\pi}\left[1 - \frac{1}{\mathcal{R}_0^{pp}}\right], \tag{37}$$

It is practically difficult to find the critical value of the vaccine efficacy ν^c in an heterogeneous population because it may depend on the conditions of manufacturing and conservation of the vaccine as well as the immune depressive status of every vaccinated individus in the host population. Also, a high efficacy vaccine leads to a lower vaccination coverage to eradicate the disease.

4. Sensitivity analysis

4.1. Local sensitivity analysis of $\mathcal{R}_0$

The local sensitivity analysis is based on the normalized sensitivity index of $\mathcal{R}_0$. The normalized forward sensitivity index of a variable to a parameter is the number of the relative change in the variable to the relative change in the parameter. Since the basic reproduction number is a differentiable function of the parameters, the sensitivity index may alternatively be defined using partial derivatives [43]. To this aim, denoting by Φ the generic parameter of system (2), we evaluate the normalized sensitivity index

$$S_\Phi = \frac{\Phi}{\mathcal{R}_0}\frac{\partial \mathcal{R}_0}{\partial \Phi},$$

which indicates how sensitive $\mathcal{R}_0$ is to a change of parameter Φ. A positive (resp. negative) index indicates that an increase in the parameter value results in an increase (resp. decrease) in the $\mathcal{R}_0$ value.
Considering the parameter values in Table 7, we tabulate the indexes of the remaining parameters in Table 2.

From Table 2, we can observe that the parameters β_v, β_e, Λ_p and ϕ respectively have a positive influence in the value of $\mathcal{R}_0$. This means that the increase or the decrease of these parameters, will increase or decrease $\mathcal{R}_0$. The indexes for parameters $\xi, \delta_p, \pi, \nu, \mu_p, H_p$ and H_e, show that increasing their values, will decrease the value of $\mathcal{R}_0$. From these analyses, it is worth remakable that a higher vaccine efficacy ν and the higher prevalence rate π decreases $\mathcal{R}_0$. Using the parameter values in Table 7, the numerical results displayed in Figure 2 illustrate the role of ν and π on the basic reproduction number $\mathcal{R}_0$, from which we observe that $\mathcal{R}_0$ decreases whenever the parameters ν and π increases. This suggests that, an optimal control measure could be the combination of the rate of vaccine efficacy and prevalence rate.

4.2. Sensitivity analysis of model's parameters

We carry out sensitivity analysis to ascertain the uncertainty of the parameters to the model output. This is vital since it enables us to identify critical output parameters. Sensitivity

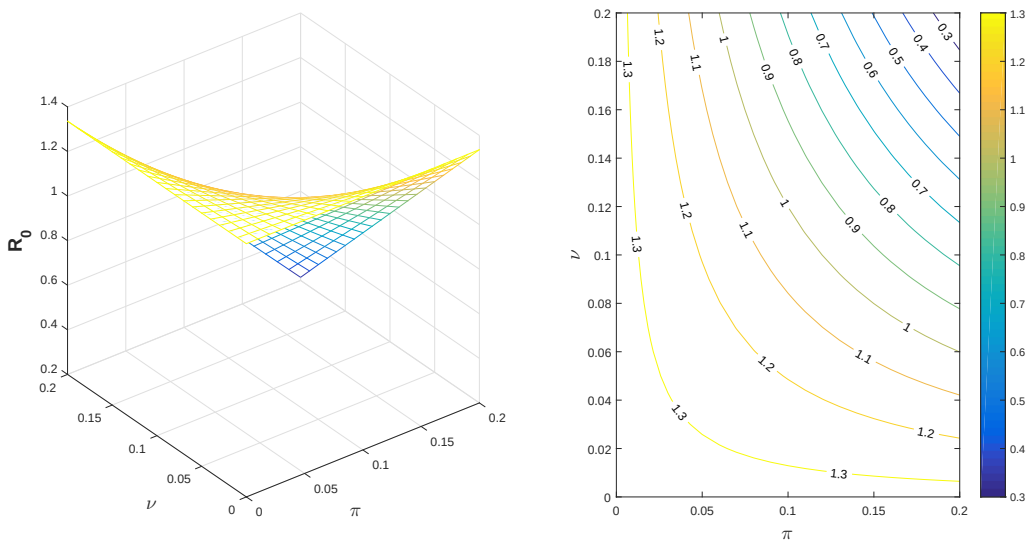

Figure 2: The basic reproduction number $\mathcal{R}_0$ plotted as function of the vaccine efficacy of poultry ν and Prevalence rate π.

and uncertainty analysis are performed using the Latin hypercube sampling scheme, a Monte Carlo stratified sampling method that allows to obtain an unbiased estimate of the model output for a given set of input parameter value. The parameter space is simultaneously sampled is used to compute unbiased estimate of output values for state variables [54, 55]. We use predefined variation of the model parameters at 10% and 50% relative to the referential values. Using algorithm from [54, 55], we compute the PRCC of parameters against model's variables $S_p, V_p, I_p, C, S_h, E_h$ and I_h. We use a sample of size 1000 to identify relationship between parameters and output variables. A positive (negative) correlation coefficient corresponds to an increasing (decreasing) monotonic trend between the model's variable and the parameter under consideration.

Note that one parameter in table 4 and 5 is said "significantly correlate to one state variable" if absolute value of PRCC is more than 0.5 and p-value less than 0.001

Table 6 present the eight most influential parameters of model system (3). According to the result obtained in table 6, the parameters $\Lambda_p, \pi, \delta_p, \beta_e, \phi, \xi, H_e$ and μ_p should significantly affect the output. Thus, the sensitivity analysis results suggest that an effective control strategy would be the implementation of mass vaccination program of the poultry population on the risks of contact transmission.

5. Analysis of model with personal protection only

The model is given by

$$\begin{cases} \dfrac{dS_p}{dt} = \Lambda_p - (\delta_p + \lambda)S_p, \\ \dfrac{dI_p}{dt} = S_p\lambda - (\delta_p + \mu_p)I_p, \\ \dfrac{dS_h}{dt} = \Lambda_h + aE_h + \gamma I_h - (1-cq)\tau_p \dfrac{S_h I_p}{H_{ph} + I_p} - (1-cq)\tau_e \dfrac{S_h C}{H_{eh} + C} - \delta_h S_h, \\ \dfrac{dE_h}{dt} = (1-cq)\tau_p \dfrac{S_h I_p}{H_{ph} + I_p} + (1-cq)\tau_e \dfrac{S_h C}{H_{eh} + C} - (a + \delta_h + \epsilon)E_h, \\ \dfrac{dI_h}{dt} = \epsilon E_h - (\gamma + \mu_h + \delta_h)I_h, \\ \dfrac{dC}{dt} = \phi I_p - \xi C. \end{cases} \tag{38}$$

where λ is the infection force defined in Equation (2).

5.1. The disease-free equilibrium and its stability

The disease-free equilibrium is $E^0_{pp} = (\frac{\Lambda_p}{\delta_p}, 0, \frac{\Lambda_h}{\delta_h}, 0, 0, 0)$. Following Van Den Driessche and Watmough[36], the basic reproduction number of model system (38) is

$$\mathcal{R}_0^{pp} = \frac{\Lambda_p(\beta_v H_e \xi + \beta_e H_p \phi)}{H_e H_p \delta_p \xi(\delta_p + \mu_p)}.$$

The relevance of the reproduction number is due to the following result established from [36, Theorem 2].

Theorem 5 *The DFE E^0_{pp} is locally asymptotically stable in Ω if $\mathcal{R}_0^{pp} < 1$ and unstable if $\mathcal{R}_0^{pp} > 1$.*

Herein, we establish the global stability of the equilibria for the continuous system (38). This is achieved by constructing Lyapunov functions. Then, we have the following results about the global stability.

Theorem 6 *The disease-free equilibrium of model system* (38) *is globally asymptotically stable (GAS) in Ω if $\mathcal{R}_0^{pp} \leq 1$.*

For the proof of Theorem 6, see Appendix C.

5.2. Endemic equilibrium and its stability

The endemic equilibrium of avian-human personal protection is $E^*_{pp} = (S^*_{pp}, I^*_{pp}, S^*_h, E^*_h, I^*_h, C^*)$. We have

$$S^*_{pp} = \frac{\Lambda_p}{\delta_p} - I^*_{pp}; \quad C^* = \frac{\phi}{\xi} I^*_{pp}, \tag{39}$$

and I^*_{pp} must satisfy the following equation:

$$b_2 {I^*_{pp}}^2 + b_1 I^*_{pp} + b_0 = 0, \tag{40}$$

where

$$\begin{aligned} b_2 &= -\frac{\phi(\delta_p+\mu_p)}{\xi}(\delta_p+\beta_v+\beta_e), \\ b_1 &= \frac{\phi\Lambda_p}{\xi}(\beta_v+\beta_e) - (\delta_p+\mu_p)\left[H_e\beta_v + \frac{\beta_e H_p\phi}{\xi} + \frac{H_p\delta_p\phi}{\xi} + H_e\delta_p\right], \\ b_0 &= \frac{\Lambda_p(\beta_v\xi H_e + \beta_e\phi H_p)}{\xi} - \delta_p(\delta_p+\mu_p)H_eH_p = H_pH_e\delta_p(\delta_p+\mu_p)\left(\mathcal{R}_0^{pp}-1\right). \end{aligned}$$

Equation (40) has a unique positive solution if $\mathcal{R}_0^{pp} > 1$ and no positive solution whenever $\mathcal{R}_0^{pp} \leq 1$. Substituting this solution by its value in (33), we have the positivity and uniqueness of (32). These investigations are summarized in the following result.

Lemma 2 *The model (38) has:*

1. *a unique endemic equilibrium whenever $\mathcal{R}_0^{pp} > 1$;*
2. *no endemic equilibrium whenever $\mathcal{R}_0^{pp} \leq 1$.*

Now, we investigate the stability of the unique endemic equilibrium E^*_{pp} when $\mathcal{R}_0^{pp} > 1$. To do this, we use the method based on Volterra-Lyapunov stable matrices. We have obtained the following result.

Theorem 7 *The positive endemic equilibrium E^*_{pp} of model (38) is globally asymptotically stable when $\mathcal{R}_0^{pp} > 1$.*

The proof of Theorem 7 is given in Appendix D.

5.3. Impact of personal protection

Personal protection is applied in the event of a pandemic (when $\mathcal{R}_0^{pp} \geq 1$). Even through $\dfrac{(1-cq)\Lambda_h\tau_p}{H_{ph}\delta_h(\delta_p+\mu_p)} + \dfrac{(1-cq)\Lambda_h\tau_e\phi}{H_{eh}\delta_h\xi(\delta_p+\mu_p)}$ is not the basic reproduction number $\mathcal{R}_0^{pp}$, we let $\mathcal{R}_{pp} = \dfrac{(1-cq)\Lambda_h\tau_p}{H_{ph}\delta_h(\delta_p+\mu_p)} + \dfrac{(1-cq)\Lambda_h\tau_e\phi}{H_{eh}\delta_h\xi(\delta_p+\mu_p)}$, to examine the effect of c and q on the disease in the human population. To do so, we find the minimum values of τ_p and τ_e by using the fact that $\mathcal{R}_{pp} > 1$.

$$\mathcal{R}_{pp} > 1 \Leftrightarrow \frac{(1-cq)\Lambda_h\tau_p}{H_{ph}\delta_h(\delta_p+\mu_p)} > 1 \text{ or } \frac{(1-cq)\Lambda_h\tau_e\phi}{H_{eh}\delta_h\xi(\delta_p+\mu_p)} > 1.$$

Let

$$\tau_p^c = \frac{H_{ph}\delta_h(\delta_p+\mu_p)}{\Lambda_h} \text{ and } \tau_e^c = \frac{H_{eh}\delta_h\xi(\delta_p+\mu_p)}{\Lambda_h\phi}.$$

The constraint $\mathcal{R}_{pp} > 1$ defines implicitly two critical values $\tau_p > \tau_p^c/(1-cq)$ and $\tau_e > \tau_e^c/(1-cq)$ that must be achieved for reduce the infection.

6. Numerical studies

In this section, we present some numerical simulations to investigate the spread of avian influenza. The parameters are fixed in the table 7.

6.1. General dynamics

Figure 3 is an illustration of Theorem 3, showing the GAS of disease-free equilibrium Q^0 of model system (3) using various initial condition when $\xi = 2000$ (so that $\mathcal{R}_0 = 0.9888$). All other parameter values are as in table 7. It illustrates that the disease disappears in host populations when $\mathcal{R}_0 \leq 1$.

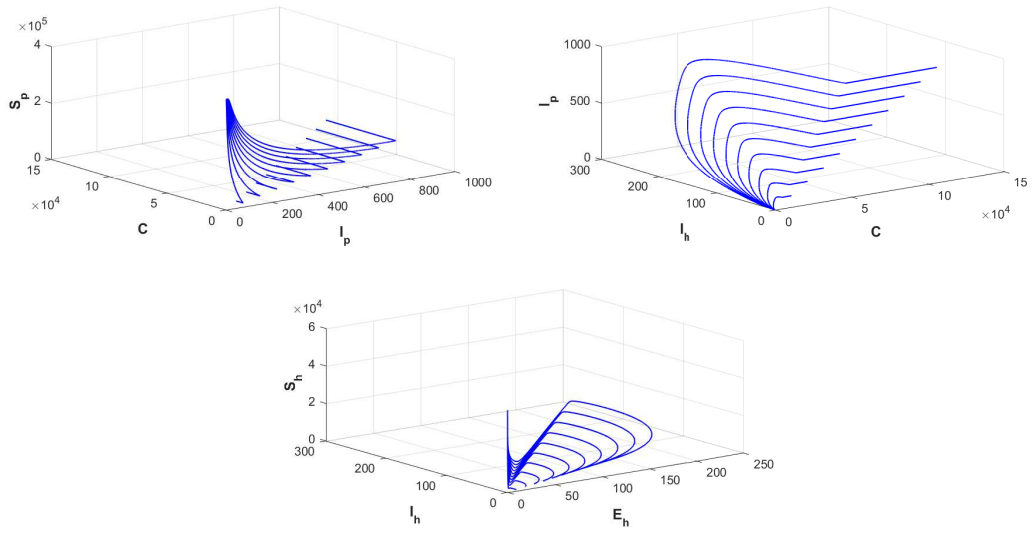

Figure 3: Global stability of disease-free equilibrium Q^0 (Theorem 3).

Figure 4 shows the stability of the endemic equilibrium Q^* of model system (3) as demonstrated in Theorem 4 when $\xi = 1700$ (so that $\mathcal{R}_0 = 1.1633$). All other parameter values are as in table 7. Although the stability of the endemic equilibrium have been established analytically in a neighborhood of $\mathcal{R}_0 = 1$, numerical simulation show that the endemic equilibrium is stable over a wide range of values of $\mathcal{R}_0 > 1$.

6.2. Effect of vaccination and personal protection

Now, numerical simulations are carried out to investigated the impact of poultry vaccination and the effet of personal protection on the dynamical transmission of AI within a human community. In all simulations, models systems (3) and (38) was simulated with the following initial conditions which has been choosen arbitrarily: $S_p^0 = 475960, V_p^0 =$

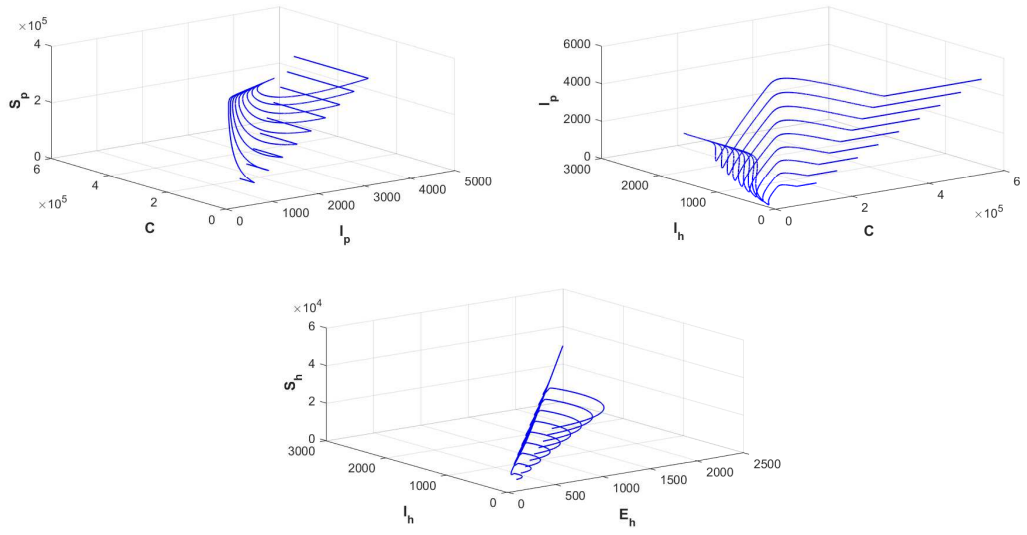

Figure 4: Stability of endemic equilibrium Q^* (Theorem 4).

$244040, I_p^0 = 10000, S_h^0 = 58500, E_h^0 = 1000, I_h^0 = 1000$, and $C^0 = 110000$. Results of numerical simulations are despited in Figure 5, Figure 6 , Figure 7 and Figure 8.

Case 1: Here we numerically investigate the effect of the critical values of π and ν on the AI dynamical transmission model (3). The time evolution of infected individuals in an outbreak with 55% of vaccine efficacy $(\nu = 0.55)$ (so that $\pi^c = 0.3895$) for three different values of proportion of susceptible population vaccinated $\pi : \pi = 0$ (so that $\mathcal{R}_0 = \mathcal{R}_0^{pp} = 1.2726$), $\pi = 0.3$ (so that $\mathcal{R}_0 = 1.0626$ and $\pi < \pi^c$) and $\pi = 0.5$ (so that $\mathcal{R}_0 = 0.9226$ and $\pi > \pi^c$) is depicted in Figure 5. All other parameter values are as in Table 7. It is evident that a large coverage of vaccination may dramatically decrease the number of infected individuals. This implies that the condition $\pi > \pi^c$ is necessary but not sufficient for the eradication of the disease within a community.

Figure 6 presents the time evolution of infected individuals in an outbreak considering that 30% of the population of susceptible is vaccinated (i.e. $\pi = 0.3$) (so that $\nu^c = 0.7140$) for three different values of the efficacy level: $\nu = 0$ (so that $\mathcal{R}_0 = \mathcal{R}_0^{pp} = 1.2726$), $\nu = 0.5$ (so that $\mathcal{R}_0 = 1.0817$ and $\nu < \nu^c$) and $\nu = 0.8$ (so that $\mathcal{R}_0 = 0.9672$ and $\nu > \nu^c$). All other parameter values are as in Table 7. It illustrates that the production of vaccine with a high level of efficacy has a preponderant role in the reduction of the disease spread.

Case 2: Figure 7 illustrates this statement. When 90% of the human population engaged in personal protection, the population of susceptible humans increases while the population of infected humans decreases as q increase. Aside from implamentating personal protection and ensuring its efficacy, it is also important that the strategy is employed by a huge percentage of population to be able to effectively decrease the number of infected humans.

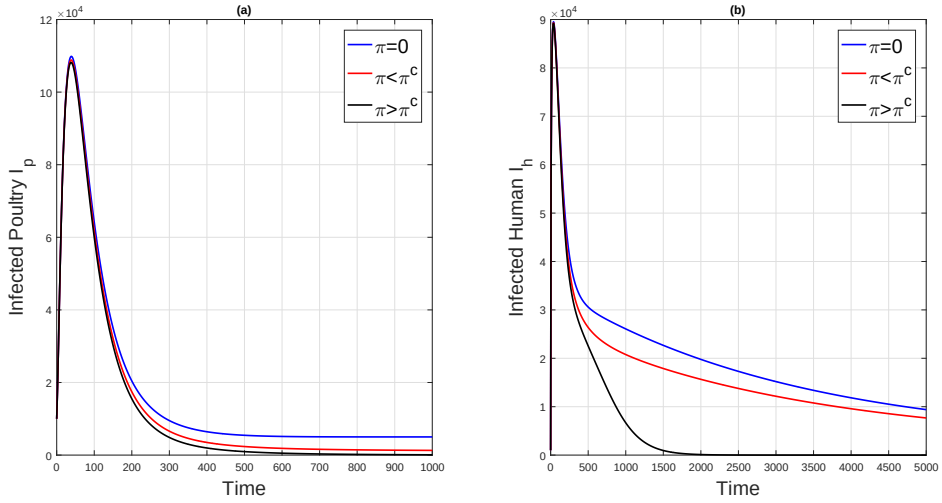

Figure 5: Infected poultry and human for three different values of proportion of susceptible poultry population vaccinated π when $\nu = 0.55$, $\xi = 2100$, $c = q = 0$ (so that $\pi^c = 0.3895$). (a) Infected poultry and (b) infected human. All other parameter values are as in Table 7.

Case 3: Figure 8 shows that by employing non-pharmaceutical interventions (personal protection), we will only be able to reduce the level of endemicity of the disease in the human population. So the disease cannot be eradicated. On the other hand, a pharmaceutical control strategy (vaccination) will make it possible to eradicate the disease even if this will take place over time. In conclusion, the pharmaceutical control strategy (vaccination) is more effective than personal protection in combating the disease. Shortly, the combination of these two control strategies will be essential if we want to eradicate the disease in a shorter time.

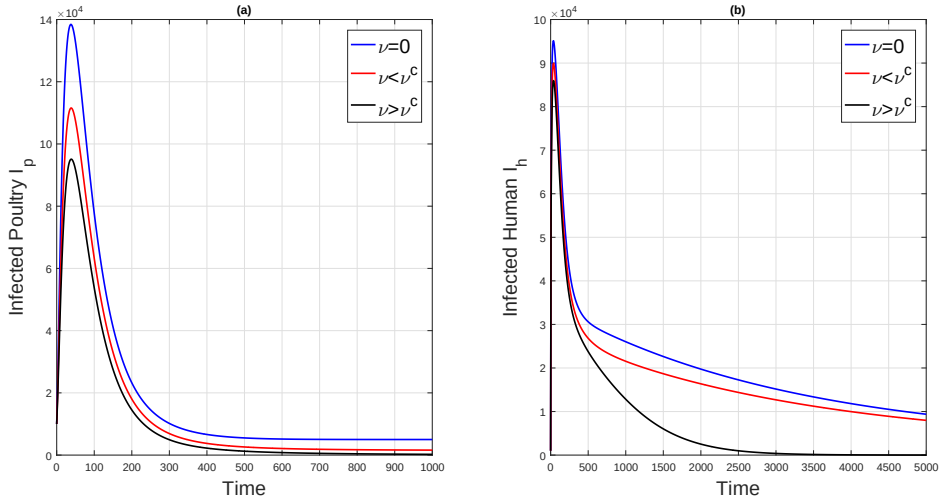

Figure 6: Infected poultry and human for three different values of the efficacy level of vaccine ν when $\pi = 0.3$, $\xi = 2100$, $c = q = 0$ (so that $\nu^c = 0.7140$). (a) Infected poultry and (b) infected human. All other parameter values are as in Table 7.

7. Conclusion

In this paper, we have formulated a mathematical model for the dynamical transmission of avian influenza A (H7N9) in which the following factors are incorporated: (i) vaccination against avian influenza A, (ii) waning of vaccination, (iii) efficacy of vaccine and (iv) the efficiency of personal protection. A qualitative analysis of the model has been presented.

Our main findings on the long-term dynamics of the system can be summarized as follows:

(1) We computed the disease-free equilibium and derived the basic reproduction number $\mathcal{R}_0$ that determines the outcome of avian influenza A within the community.

(2) We proved that the disease-free equilibrium is globally asymptotically stable whenever $\mathcal{R}_0 \leq 1$ on a positively invariant region.

(3) We showed that the model has a unique endemic equilibrium when $\mathcal{R}_0 > 1$. We also established the local asymptotic stability of this unique endemic equilibrium when $\mathcal{R}_0 > 1$ but close to 1 and the global asymptotic stability of the endemic equilibrium when $\mathcal{R}_0 > 1$. A way of distributing the vaccines to poultry against avian influenza A or employing personal protection as well as their features and some of coverage thresholds were introduced in oder to study the effect of the vaccine coverage, vaccine efficacy and the efficiency of personal protection. The main goal of a vaccination program and employment of personal protection is to reduce the prevalence of the disease and ultimately to eradicate it. It was shown that short-term eradication succes depends on the type of vaccine as well as on the vaccination coverage,

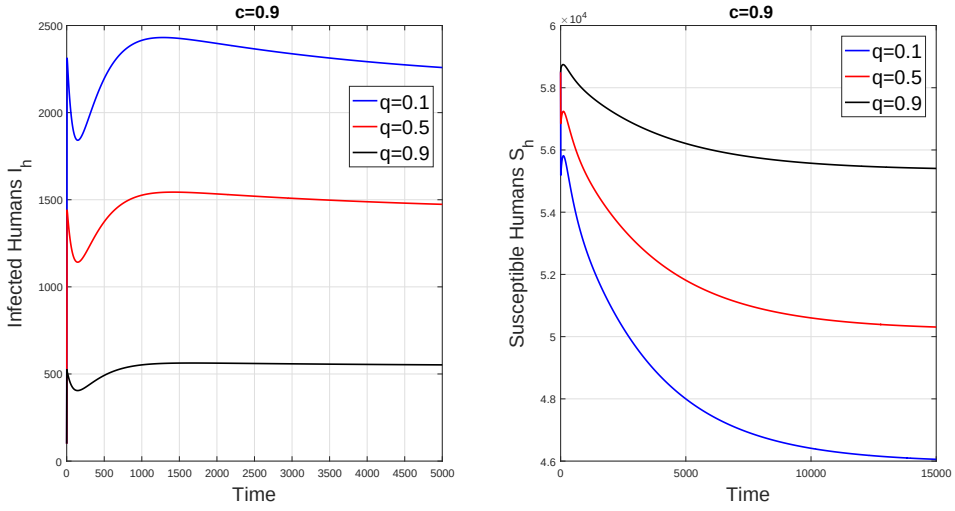

Figure 7: State trajectories of human personal protection model. When $\xi = 2500$ and all other parameter values are as in Table 7.

percentage of human population employing personal protection and the efficiency of the personal protection.

(4) The sensitivity analysis of the threshold number $\mathcal{R}_0$ and of the model has been investigated. We found that for the threshold number $\mathcal{R}_0$, an effective control strategy would be the implementation of mass vaccination program in the poultry population for the risks of contact transmission of avian influenza to human population. However, we found that the model variables are most sensitives to the prevalence rate of the vaccination program, indirect transmission rate in poultry, natural death rate of poultry, avian influenza induced mortality to poultry, emission rate of virus by poultry and degradation rate of virus. Therefore, in an epidemic situation, it is urgent to sensibilize human population about the risks of transmission of avian influenza A through contact with poultry or poultry environment and to take on charge vaccination program (poultry vaccination) and barrier measures for the human population (personal protection).

(5) Numerical results have been presented to illustrate and validate theoritical results. Through numerical simulations, we found that the best way to control the transmission or to fight an avian influenza outbreak is to combine a non-medicinal (personal protection) and medicinal (vaccination) control strategies.

Acknowledgment

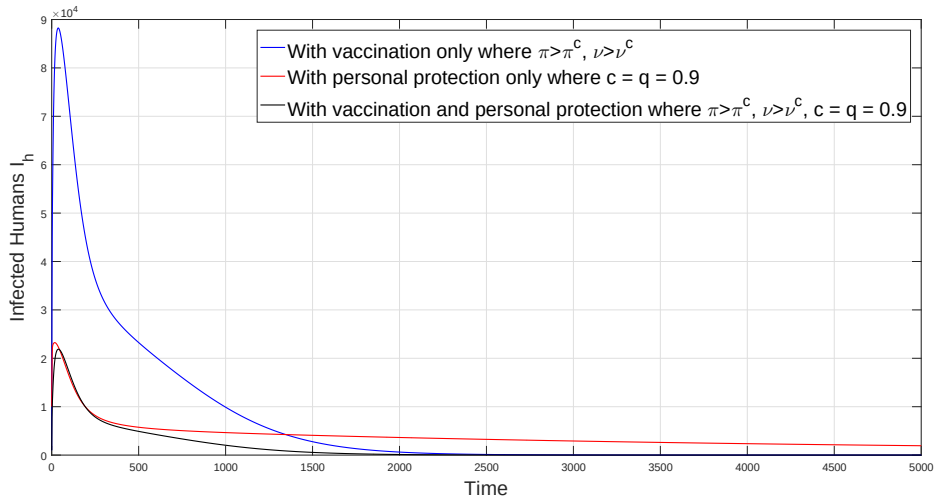

Figure 8: Comparison between the suggested control strategies.

A. Proof of the condition $\alpha(\overline{A}_2) \leq 0 \Leftrightarrow \mathcal{R}_0 \leq 1$.

Herein, we show that the condition $\alpha(\overline{A}_2) \leq 0$ is equivalent to $\mathcal{R}_0 \leq 1$. To check condition H_5 of Theorem from Kamgang and Sallet [59], we will use the following Lemma:

Lemma 3 *Let M be a square Metzler matrix written in block form $M = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$. where $\mathcal{A}$ and $\mathcal{D}$ are square matrices. Then the matrix M is Metzler stable if and only if matrices $\mathcal{A}$ and $\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}$ are Metzler stable.*

The matrix $\overline{A}_2$ can be expressed in the form of the matrix M with

$$\mathcal{A} = \begin{bmatrix} \dfrac{\beta_v \left[S_p^0 + (1-\nu)V_p^0\right]}{H_p} - (\delta_p + \mu_p) & 0 \\ \dfrac{(1-cq)\tau_p}{H_{ph}} S_h^0 & -(a + \epsilon + \delta_h) \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0 & \dfrac{\beta_e \left[S_p^0 + (1-\nu)V_p^0\right]}{H_e} \\ 0 & \dfrac{(1-cq)\tau_e}{H_{eh}} S_h^0 \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} 0 & \epsilon \\ \phi & 0 \end{bmatrix} \text{ and } \mathcal{D} = \begin{bmatrix} -(\gamma + \mu_h + \delta_h) & 0 \\ 0 & -\xi \end{bmatrix}.$$

The matrix $\mathcal{A}$ is Metzler stable if and only if $\mathcal{R}_0 \leq 1$. Indeed

$$\frac{\beta_v \left[S_p^0 + (1-\nu)V_p^0\right]}{H_p} - (\delta_p + \mu_p) = (\delta_p + \mu_p)(\mathcal{R}_0 - 1) - \frac{\beta_e \phi}{H_e \xi} \left[S_p^0 + (1-\nu)V_p^0\right].$$

A simple calculation yields

$$\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B} = \begin{bmatrix} -(\gamma + \delta_h + \mu_h) & -\dfrac{\beta_e \xi}{H_e} \left[S_p^0 + (1-\nu)V_p^0\right] a_{21} - \dfrac{\epsilon(1-cq)\tau_e}{H_{eh}} S_h^0 a_{22} \\ 0 & -\xi - \dfrac{\beta_e \phi}{H_e} \left[S_p^0 + (1-\nu)V_p^0\right] a_{11} \end{bmatrix},$$

where

$$a_{11} = -\frac{(a+\epsilon+\delta_h)}{\det \mathcal{A}},\ a_{21} = -\frac{(1-cq)\tau_p S_h^0}{H_{ph}\det \mathcal{A}},\ a_{22} = \frac{\beta_v\left[S_p^0+(1-\nu)V_p^0\right]-H_p(\delta_p+\mu_p)}{H_p \det \mathcal{A}},$$

$$\det \mathcal{A} = -\frac{\beta_v(a+\epsilon+\delta_h)\left[S_p^0+(1-\nu)V_p^0\right]}{H_p}+(a+\epsilon+\delta_h)(\delta_p+\mu_p).$$

The matrix $\mathcal{D}-\mathcal{C}\mathcal{A}^{-1}\mathcal{B}$ is Metzler stable if and only if

$$\xi - \frac{\dfrac{\beta_e\phi\left[S_p^0+(1-\nu)V_p^0\right]}{H_e}}{(\delta_p+\mu_p)-\dfrac{\beta_v\left[S_p^0+(1-\nu)V_p^0\right]}{H_p}} \geq 0.$$

That is,

$$\frac{\beta_e\phi\left[S_p^0+(1-\nu)V_p^0\right]}{H_e} \leq \xi(\delta_p+\mu_p)-\frac{\beta_v\xi\left[S_p^0+(1-\nu)V_p^0\right]}{H_p} \Leftrightarrow \mathcal{R}_0 \leq 1.$$

B. Proof of Theorem 4

Proof. To apply this theory, we first rename the state variables. Let $z_1 = S_p, z_2 = V_p, z_3 = I_p, z_4 = S_h, z_5 = E_h, z_6 = I_h$ *and* $z_7 = C$ so that $N_p = z_1+z_2+z_3$, $N_h = z_4+z_5+z_6$. Further, by using the vector notation $z = (z_1, z_2, z_3, z_4, z_5, z_6, z_7)^T$, the Avian influenza model (3) can be written in the form $\dot{z} = f(z)$, with $f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T$ as follows:

$$\begin{cases} \dot{z}_1 = f_1 = (1-\pi)\Lambda_p - [\lambda+\delta_p]\, z_1, \\ \dot{z}_2 = f_2 = \pi\Lambda_p - [(1-\nu)\lambda+\delta_p]\, z_2, \\ \dot{x}_3 = f_3 = \lambda\left[z_1+(1-\nu)z_2\right] - (\delta_p+\mu_p)z_3, \\ \dot{x}_4 = f_4 = \Lambda_h + az_5 + \gamma z_6 - (1-cq)\tau_p\dfrac{z_4z_3}{H_{ph}+z_3} - (1-cq)\tau_e\dfrac{z_4z_7}{H_{eh}+z_7} - \delta_h z_4, \\ \dot{x}_5 = f_5 = (1-cq)\tau_p\dfrac{z_4z_3}{H_{ph}+z_3} + (1-cq)\tau_e\dfrac{z_4z_7}{H_{eh}+z_7} - (a+\delta_p+\epsilon)z_5, \\ \dot{x}_6 = f_6 = \epsilon z_5 - (\gamma+\mu_p+\delta_p)z_6, \\ \dot{x}_7 = f_7 = \phi z_3 - \xi z_7. \end{cases} \tag{41}$$

System (41) has a DFE given by $Q^0 = (S_p^0, V_p^0, 0, S_h^0, 0, 0, 0)$. The Jacobian of system (41) at the DFE, is the same as for the one in proof of Theorem 2. The basic reproduction number of the transformed (linearized) model system (41) is the same as that of the original model (3).

Let $\sigma_e > 0$ be the non-negative real numbers such that $\beta_e = \sigma_e\beta_v$, then the basic reproduction number $\mathcal{R}_0$ becomes

$$\mathcal{R}_0 = \frac{\beta_e H_e\xi + \beta_e\phi H_p\sigma_e}{H_eH_p\sigma_e\xi(\delta_p+\mu_p)}\left[S_p^0+(1-\nu)V_p^0\right].$$

Therefore, choosing β_e as a bifurcation parameter, by solving for β_e when $\mathcal{R}_0 = 1$, we obtain:

$$\beta_e = \beta_e^* = \frac{H_e H_p \sigma_e \xi(\delta_p + \mu_p)}{(H_e\xi + \phi H_p \sigma_e)\left[S_p^0 + (1-\nu)V_p^0\right]}.$$

It follows that the Jacobian $(J|_{Q^0})$ of system (41) at the DFE Q^0, with $\beta_e = \beta_e^*$, denoted by $J|_{\beta_e^*}$ has a simple zero eigenvalue (with all other eigenvalues having negative real parts). Hence, the Centre Manifold theory [50] can be used to analyze the dynamics of system (41). In particular, [50, Theorem 4.1], will be used to show that, when $\mathcal{R}_0 > 1$, there exists a unique endemic equilibrium of system (41) (as shown in Lemma 1) which is locally asymptotically stable for $\mathcal{R}_0$ near 1, under certain conditions. In order to apply the above theorem, the following computations are necessary (it should be noted that we are using β_e^* as the bifurcation parameter, in place of ϕ [50, Theorem 4.1]).

Eigenvectors of $J|_{\beta_e^*}$: The right eigenvector corresponding to the zero eigenvalue is:

$$u = (u_1, u_2, u_3, u_4, u_5, u_6, u_7)^T.$$

By solving the system

$$\begin{cases} -\delta_p u_1 - \dfrac{\beta_e^*}{H_p\sigma_e} S_p^0 u_3 - \dfrac{\beta_e^*}{H_e} S_p^0 u_7 = 0, \\ -\delta_p u_2 - \dfrac{(1-\nu)\beta_e^*}{H_p\sigma_e} V_p^0 u_3 - \dfrac{(1-\nu)\beta_e^*}{H_e} V_p^0 u_7 = 0, \\ \left[\dfrac{\beta_e^*}{H_p\sigma_e}\left[S_p^0 + (1-\nu)V_p^0\right] - (\delta_p + \mu_p)\right] u_3 + \dfrac{\beta_e^*}{H_e}\left[S_p^0 + (1-\nu)V_p^0\right] u_7 = 0, \\ -\dfrac{(1-cq)\tau_p S_h^0}{H_{ph}} u_3 - \delta_h u_4 + a u_5 + \gamma u_6 - \dfrac{(1-cq)\tau_e S_h^0}{H_{eh}} u_7 = 0, \\ \dfrac{(1-cq)\tau_p S_h^0}{H_{ph}} u_3 - (a + \epsilon + \delta_h) u_5 + \dfrac{(1-cq)\tau_e S_h^0}{H_{eh}} u_7 = 0, \\ \epsilon u_5 - (\gamma + \delta_h + \mu_h) u_6 = 0, \\ \phi u_3 - \xi u_7 = 0, \end{cases}$$

yields

$$u_7 = \frac{\phi}{\xi} u_3 \ , \ u_3 = u_3 > 0 \ , \ u_4 = -\frac{(1-cq)S_h^0}{\delta_h}\left[\frac{\tau_p}{H_{ph}} + \frac{\tau_e\phi}{H_{eh}\xi}\right] + \left[\frac{a}{\delta_h} + \frac{\gamma\epsilon}{\delta_h(\gamma + \delta_h + \mu_h)}\right] u_5,$$

$$u_5 = \frac{(1-cq)S_h^0}{a + \epsilon + \delta_h}\left[\frac{\tau_p}{H_{ph}} + \frac{\tau_e\phi}{H_{eh}\xi}\right] u_3 \ , \ u_6 = \frac{\epsilon(1-cq)S_h^0}{(a + \epsilon + \delta_h)(\gamma + \delta_h + \mu_h)}\left[\frac{\tau_p}{H_{ph}} + \frac{\tau_e\phi}{H_{eh}\xi}\right] u_3,$$

$$u_1 = -\frac{(1-\nu)\beta_e^* V_p^0}{\delta_p}\left[\frac{1}{H_p\sigma_e} + \frac{\phi}{H_e\xi}\right] u_3, \ u_2 = -\frac{\beta_e^* S_p^0}{\delta_p}\left[\frac{1}{H_p\sigma_e} + \frac{\phi}{H_e\xi}\right] u_3.$$

Similarly, the components of the left eigenvectors (corresponding to the zero eigenvalue), denoted by

$$v = (v_1, v_2, v_3, v_4, v_5, v_6, v_7),$$

is obtained by solving the system

$$\begin{cases} \left[\dfrac{\beta_e^*}{H_p\sigma_e}\left[S_p^0+(1-\nu)V_p^0\right]-(\delta_p+\mu_p)\right]v_3+\dfrac{(1-cq)\tau_pS_h^0}{H_{ph}}v_5+\phi v_7=0,\\ v_1=v_2=v_4=0,\\ -(a+\epsilon+\delta_h)v_5+\epsilon v_6=0,\\ -(\gamma+\delta_h+\mu_h)v_6=0,\\ \dfrac{\beta_e^*}{H_e}\left[S_p^0+(1-\nu)V_p^0\right]v_3+\dfrac{(1-cq)\tau_eS_h^0}{H_{eh}}v_5-\xi v_7=0. \end{cases}.$$

Hence,

$$v_1=0\ ,\ v_3=\frac{H_e\xi}{\beta_e^*\left[S_p^0+(1-\nu)V_p^0\right]}v_7\ ,\ v_4=0\ ,\ v_5=0\ ,\ v_6=0\ ,\ v_7=v_7>0.$$

Computation of a: For system (41), the corresponding non-zero partial derivatives of $f_i\ (i=1,2,3,4,5,6,7)$ calculated at the disease free equilibrium are given by:

$$\frac{\partial^2 f_3}{\partial x_3^2}=-2\frac{\beta_e^*}{H_p^2\sigma}\left[S_p^0+(1-\nu)V_p^0\right]\ ,\ \frac{\partial^2 f_3}{\partial x_7^2}=-2\frac{\beta_e^*}{H_e^2}\left[S_p^0+(1-\nu)V_p^0\right],$$

$$\frac{\partial^2 f_3}{\partial x_1\partial x_3}=\frac{\beta_e^*}{H_p\sigma}\ ,\ \frac{\partial^2 f_3}{\partial x_1\partial x_7}=\frac{\beta_e^*}{H_e}\ ,\ \frac{\partial^2 f_3}{\partial x_2\partial x_3}=\frac{(1-\nu)\beta_e^*}{H_p\sigma}\ ,\ \frac{\partial^2 f_3}{\partial x_2\partial x_7}=\frac{(1-\nu)\beta_e^*}{H_e}.$$

Consequently, we calculate the associated bifurcation coefficient **a**

$$\begin{aligned} \mathbf{a} &= \sum_{k,i,j=1}^{7} v_k u_i u_j \frac{\partial^2 f_k}{\partial x_i\partial x_j}(Q^0),\\ &= v_3\left(u_3^2\frac{\partial^2 f_3}{\partial x_3^2}+u_7^2\frac{\partial^2 f_3}{\partial x_7^2}+2u_1u_3\frac{\partial^2 f_3}{\partial x_1\partial x_3}+2u_1u_7\frac{\partial^2 f_3}{\partial x_1\partial x_7}+2u_1u_3\frac{\partial^2 f_3}{\partial x_2\partial x_3}+2u_1u_7\frac{\partial^2 f_3}{\partial x_2\partial x_7}\right)<0. \end{aligned}$$

Computation of b: For system (41), the corresponding non-zero partial derivatives of $f_i,(i=1,2,3,4,5,6,7)$ calculated at the disease free equilibrium are given by:

$$\frac{\partial^2 f_3}{\partial x_3\partial\beta_e^*}=\frac{1}{H_p\sigma_e}\left[S_p^0+(1-\nu)V_p^0\right]\ \text{and}\ \frac{\partial^2 f_3}{\partial x_7\partial\beta_e^*}=\frac{1}{H_e}\left[S_p^0+(1-\nu)V_p^0\right].$$

We compute the associated bifurcation coefficient **b**

$$\begin{aligned} \mathbf{b} &= \sum_{k,i=1}^{7} v_k u_i \frac{\partial^2 f_k}{\partial x_i\partial\beta_e^*}(Q^0),\\ &= \frac{H_e\xi}{\beta_e^*\left[S_p^0+(1-\nu)V_p^0\right]}\left[\frac{1}{H_p\sigma_e}\left[S_p^0+(1-\nu)V_p^0\right]+\frac{\phi}{H_e\xi}\left[S_p^0+(1-\nu)V_p^0\right]\right]v_7u_3,\\ &= \frac{H_e\xi}{\beta_e^*}\left[\frac{1}{H_p\sigma_e}+\frac{\phi}{H_e\xi}\right]v_7u_3>0. \end{aligned}$$

Thus, the bifurcation coefficient **a** is always negative. Furthermore, the bifurcation coefficient **b** is always positive. Hence, it follows from [50, Theorem 4.1], that model (41) does undergo the transcritical forward bifurcation at $\mathcal{R}_0=1$. ■

C. Proof of Theorem 6

Proof. We use the Lyapunov function approach. Define

$$L(S_p, I_p, S_h, E_h, I_h, C) = \left[\frac{1}{\delta_p+\mu_p} + \frac{\phi}{\xi(\delta_p+\mu_p)}\right] I_p(t) + \frac{1}{\xi}C(t).$$

Then,

$$\begin{aligned}
\frac{dL}{dt} &= \left[\frac{1}{\delta_p+\mu_p} + \frac{\phi}{\xi(\delta_p+\mu_p)}\right]\frac{dI_p}{dt} + \frac{1}{\xi}\frac{dC}{dt},\\
&= \left[\frac{1}{\delta_p+\mu_p} + \frac{\phi}{\xi(\delta_p+\mu_p)}\right]\left[\frac{\beta_v S_p I_p}{H_p+I_p} + \frac{\beta_e S_p C}{H_e+C} - (\delta_p+\mu_p)I_p\right] + \frac{1}{\xi}(\phi I_p - \xi C),\\
&= \left[\frac{1}{\delta_p+\mu_p} + \frac{\phi}{\xi(\delta_p+\mu_p)}\right]\left[\frac{\beta_v S_p I_p}{H_p+I_p} + \frac{\beta_e S_p C}{H_e+C}\right] - I_p - C,\\
&= \left[\frac{\mathcal{R}_0^{pp} H_p}{\beta_v S_p^0} + \frac{\mathcal{R}_0^{pp} H_e}{\beta_e S_p^0} - \frac{\beta_e \phi H_p}{H_e \xi \beta_v(\delta_p+\mu_p)} - \frac{H_e\beta_v}{\beta_e H_p(\delta_p+\mu_p)}\right]\left[\frac{\beta_v S_p I_p}{H_p+I_p} + \frac{\beta_e S_p C}{H_e+C}\right] - I_p - C.
\end{aligned}$$

Direct calculations lead to

$$\begin{aligned}
\frac{dL}{dt} &\leq \left[\frac{\mathcal{R}_0^{pp} H_p}{\beta_v S_p^0} + \frac{\mathcal{R}_0^{pp} H_e}{\beta_e S_p^0} - \frac{\beta_e \phi H_p}{H_e \xi \beta_v(\delta_p+\mu_p)} - \frac{H_e\beta_v}{\beta_e H_p(\delta_p+\mu_p)}\right]\left[\frac{\beta_v S_p I_p}{H_p+I_p} + \frac{\beta_e S_p C}{H_e+C}\right]\\
&\quad + \frac{\beta_e \phi S_p^0 I_p}{H_e\xi(\delta_p+\mu_p)} + \frac{H_e\beta_v^2 S_p^0 I_p}{H_p^2\beta_e(\delta_p+\mu_p)} + \frac{\beta_e^2\phi H_p S_p^0 C}{H_e^2\xi\beta_v(\delta_p+\mu_p)} + \frac{\beta_v S_p^0 C}{H_p(\delta_p+\mu_p)} - \frac{H_e\beta_v\mathcal{R}_0^{pp} I_p}{H_p\beta_e}\\
&\quad - \frac{H_p\beta_e\mathcal{R}_0^{pp} C}{H_e\beta_v} - I_p - C - \mathcal{R}_0^{pp} I_p + \mathcal{R}_0^{pp} I_p - \mathcal{R}_0^{pp} C + \mathcal{R}_0^{pp} C,\\
&\leq (\mathcal{R}_0^{pp}-1)(I_p+C) - \frac{\mathcal{R}_0^{pp} H_p I_p}{S_p^0}\left[\frac{S_p^0}{H_p} - \frac{S_p}{H_p+I_p}\right] - \frac{H_e\mathcal{R}_0^{pp}\beta_v I_p}{\beta_e S_p^0}\left[\frac{S_p^0}{H_p} - \frac{S_p}{H_p+I_p}\right]\\
&\quad + \frac{\beta_e\phi H_p I_p}{H_e\xi(\delta_p+\mu_p)}\left[\frac{S_p^0}{H_p} - \frac{S_p}{H_p+I_p}\right] + \frac{H_e\beta_v^2 I_p}{H_p\beta_e(\delta_p+\mu_p)}\left[\frac{S_p^0}{H_p} - \frac{S_p}{H_p+I_p}\right]\\
&\quad - \frac{H_p\beta_e\mathcal{R}_0^{pp} C}{\beta_v S_p^0}\left[\frac{S_p^0}{H_e} - \frac{S_p}{H_e+C}\right] - \frac{H_e\mathcal{R}_0^{pp} C}{S_p^0}\left[\frac{S_p^0}{H_e} - \frac{S_p}{H_e+C}\right]\\
&\quad + \frac{\beta_e^2\phi H_p C}{H_e\xi\beta_v(\delta_p+\mu_p)}\left[\frac{S_p^0}{H_e} - \frac{S_p}{H_e+C}\right] + \frac{H_e\beta_v}{H_p(\delta_p+\mu_p)}\left[\frac{S_p^0}{H_e} - \frac{S_p}{H_e+C}\right].
\end{aligned}$$

Finally,

$$\frac{dL}{dt} \leq (\mathcal{R}_0^{pp}-1)(I_p+C) - \frac{(\xi+\phi)(S_p^0-S_p)}{\xi(\delta_p+\mu_p)}\left[\frac{\beta_v I_p}{H_p} + \frac{\beta_e C}{H_e}\right].$$

Since $S_p \leq S_p^0$, we have $\dfrac{dL}{dt} \leq 0$, whenever $\mathcal{R}_0^{pp} \leq 1$.

Moreover, $\dfrac{dL}{dt} = 0, \Leftrightarrow I_p = C = 0$ or $S_p = S_p^0$ and $\mathcal{R}_0^{pp} = 1$.

Thus, the largest invariant set $\mathcal{H}$ such that $\mathcal{H} \subset \left\{(S_p, I_p, S_h, E_h, I_h, C) \in \mathbb{R}_+^6 / dL/dt = 0\right\}$ is $\{E_{pp}^0\}$ because in $\mathcal{H}$ one has $\lim_{t\to+\infty} I_p(t) = \lim_{t\to+\infty} C(t) = 0$. In system (38), we obtain

$\lim_{t\to+\infty} S_p(t) = S_p^0, \lim_{t\to+\infty} S_h(t) = S_h^0, \lim_{t\to+\infty} E_h(t) = \lim_{t\to+\infty} I_h(t) = 0$. By LaSalle's Invariance Principle [37], $\{E_{pp}^0\}$ is globally asymptotically stable. The proof is complete. ■

D. Proof of Theorem 7

Proof. As for the proof of the GAS of the endemic equilibrium E_{pp}^*, one should notice that, since the poultry sub-model is independent of the human population variables (S_h, E_h, I_h), system (38) takes the triangular form

$$\begin{cases} \dfrac{dx}{dt} = f(x), \quad x = (S_p, I_p, C), \\ \dfrac{dy}{dt} = g(x, y), \quad y = (S_h, E_h, I_h). \end{cases} \tag{42}$$

In order to deal with the global asymptotic stability of the unique endemic equilibrium stated in Theorem 11, the following three results are instrumental.

Theorem 8 *(Vidyagasar [46])*
Consider a $\mathcal{C}^1$ class system with an equilibrium point $(x^*; y^*)$.

$$\begin{cases} \dfrac{dx}{dt} = f(x), \\ \dfrac{dy}{dt} = g(x, y), \qquad x \in \mathbb{R}^n, y \in \mathbb{R}^m, \\ f(x^*) = 0, g(x^*, y^*) = 0. \end{cases} \tag{43}$$

If x^* is GAS in $\mathbb{R}^n$ for system $dx/dt = f(x)$, and if y^* is GAS in $\mathbb{R}^m$, for system $dy/dt = g(x^*; y)$, then equilibrium point $(x^*; y^*)$ is (locally) asymptotically stable for system (43). Moreover, if all the trajectories of (43) are positively bounded , then $(x^*; y^*)$ is GAS for (43).

Theorem 9 *Let H be a 2 × 2 matrix [44, 45]. Then*

$$H = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

is Volterra-Lyapunov stable if and only if $a_{11} < 0, a_{22} < 0$, and $a_{11}a_{22} - a_{12}a_{21} > 0$.

Theorem 10 *Let H be a non-singular n × n matrix, where $n \geq 2$, with inverse $H^{-1} = K$ and W a positive diagonal n × n matrix [35]. Let H^*, K^*, and W^* denote the $(n-1) \times (n-1)$ matrices obtained from H, K, and W, respectively, by deleting the last row and the last column. Then*

(i) if $WH + (WH)^T > 0$, we must have $a_{nn} > 0$, $W^*H^* + (W^*H^*)^T > 0$, and $W^*K^* + (W^*K^*)^T > 0$;

(ii) if $a_{nn} > 0, W^*H^* + (W^*H^*)^T > 0$*, and* $W^*K^* + (W^*K^*)^T > 0$*, it is possible to choose* $w_n > 0$ *such that* $WH + (WH)^T > 0$.

Following, Theorem 8, we first study the GAS of the endemic equilibrium x^* of the poultry system:

$$\frac{dx}{dt} = f(x) \equiv \begin{cases} \dfrac{dS_p}{dt} = \Lambda_p - \beta_v S_p \dfrac{I_p}{H_p + I_p} - \beta_e S_p \dfrac{C}{H_e + C} - \delta_p S_p, \\ \dfrac{dI_p}{dt} = \beta_v S_p \dfrac{I_p}{H_p + I_p} + \beta_e S_p \dfrac{C}{H_e + C} - (\delta_p + \mu_p) I_p, \\ \dfrac{dC}{dt} = \phi I_p - \xi C. \end{cases} \tag{44}$$

Now, we claim the following result.

Theorem 11 *The unique positive endemic equilibrium point* $x^* = (S^*_{pp}, I^*_{pp}, C^*)$ *of the system (44) is globally asymptotically stable if* $\mathcal{R}_0^{pp} > 1$.

Consider the following domain as a result of a nondimensionalized system (44)

$$\Omega_1 = \left\{ (S_p, I_p, C) \in \mathbb{R}^3_+ / 0 < S_p + I_p \leq \frac{\Lambda_p}{\delta_p}, C \leq \frac{\phi \Lambda_p}{\delta_p \xi} \right\}.$$

Next, construct the Lyapunov function

$$V = \omega_1 (S_p - S^*_{pp})^2 + \omega_2 (I_p - I^*_{pp})^2 + \omega_3 (C - C^*)^2, \tag{45}$$

with $\omega_1 > 0, \omega_2 > 0$ and $\omega_3 > 0$. Note that for the endemic equilibrium x^* , we have the following three equations for the nondimensionalized system:

$$\Lambda_p - \beta_v S^*_{pp} \frac{I^*_{pp}}{H_p + I^*_{pp}} - \beta_e S^*_{pp} \frac{C^*}{H_e + C^*} - \delta_p S^*_{pp} = 0, \tag{46a}$$

$$\beta_v S^*_{pp} \frac{I^*_{pp}}{H_p + I^*_{pp}} + \beta_e S^*_{pp} \frac{C^*}{H_e + C^*} - (\delta_p + \mu_p) I^*_{pp} = 0, \tag{46b}$$

$$\phi I^*_{pp} - \xi C^* = 0. \tag{46c}$$

Using (46a)-(46c), we obtain

$$\begin{aligned} \frac{dV}{dt} = \; & 2\omega_1 (S_p - S^*_{pp}) \left[-\frac{\beta_v S_p I_p}{H_p + I_p} - \frac{\beta_e S_p C}{H_e + C} - \delta_p S_p + \frac{\beta_v S^*_{pp} I^*_{pp}}{H_p + I^*_{pp}} + \frac{\beta_e S^*_{pp} C^*}{H_e + C^*} + \delta_p S^*_{pp} \right] \\ & + 2\omega_2 (I_p - I^*_{pp}) \left[\frac{\beta_v S_p I_p}{H_p + I_p} + \frac{\beta_e S_p C}{H_e + C} - (\delta_p + \mu_p) I_p - \frac{\beta_v S^*_{pp} I^*_{pp}}{H_p + I^*_{pp}} - \frac{\beta_e S^*_{pp} C^*}{H_e + C^*} + (\delta_p + \mu_p) I^*_{pp} \right] \\ & + 2\omega_3 (C - C^*) \left[\phi I_p - \xi C - \phi I^*_{pp} + \xi C^* \right]. \end{aligned}$$

The gatherings of some terms give

$$\begin{aligned}\frac{dV}{dt} &= -2\omega_1\left(\frac{\beta_v I_p}{H_p+I_p}+\frac{\beta_e S_p C}{H_e+C}+\delta_p\right)(S_p-S^*_{pp})^2\\ &\quad -2\omega_1\frac{\beta_v S^*_{pp}}{(H_p+I_p)(H_p+I^*_{pp})}(S_p-S^*_{pp})(I_p-I^*_{pp})\\ &\quad -2\omega_1\frac{H_e\beta_e S^*_{pp}}{(H_e+C)(H_e+C^*)}(S_p-S^*_{pp})(C-C^*)\\ &\quad +2\omega_2\left(\frac{\beta_v H_p S^*_{pp}}{(H_p+I_p)(H_p+I^*_{pp})}-(\delta_p+\mu_p)\right)(I_p-I^*_{pp})^2\\ &\quad +2\omega_2\left(\beta_v\frac{I_p}{H_p+I_p}+\beta_e\frac{C}{H_e+C}\right)(I_p-I^*_{pp})(S_p-S^*_{pp})\\ &\quad +2\omega_2\frac{H_e\beta_e S^*_{pp}}{(H_e+C)(H_e+C^*)}(I_p-I^*_{pp})(C-C^*)\\ &\quad +2\omega_3\phi(C-C^*)(I_p-I^*_{pp})-2\omega_3\xi(C-C^*)^2,\\ &= U(WH+H^TW)U^T,\end{aligned}$$

where $U=[S_p-S^*_{pp}, I_p-I^*_{pp}, C-C^*]$, $W=\text{diag}(\omega_1,\omega_2,\omega_3)$ and

$$H=\begin{bmatrix} -\frac{\beta_v I_p}{H_p+I_p}-\frac{\beta_e C}{H_e+C}-\delta_p & -\frac{\beta_v H_p S^*_{pp}}{(H_p+I_p)(H_p+I^*_{pp})} & -\frac{H_e\beta_e S^*_{pp}}{(H_e+C)(H_e+C^*)}\\ \frac{\beta_v I_p}{H_p+I_p}+\frac{\beta_e C}{H_e+C} & \frac{\beta_v H_p S^*_{pp}}{(H_p+I_p)(H_p+I^*_{pp})}-(\delta_p+\mu_p) & \frac{H_e\beta_e S^*_{pp}}{(H_e+C)(H_e+C^*)}\\ 0 & \phi & -\xi\end{bmatrix}. \tag{47}$$

The global asymptotic stability of x^* will be established if we can show that the matrix H defined in (47) is Volterra-Lyapunov stable [35]; that is, a positive diagonal matrix W exists such that $WH+H^TW$ is negative definite.
From (47), one can see that H is non-singular because

$$\begin{aligned}\det H &= \frac{\delta_p H_e\beta_e\phi S^*_{pp}}{(H_e+C)(H_e+C^*)}+\frac{\delta_p\beta_v\xi\delta_p S^*_{pp}}{(H_p+I_p)(H_p+I^*_{pp})}\\ &\quad -\xi(\delta_p+\mu_p)\left(\beta_v\frac{I_p}{H_p+I_p}+\beta_e\frac{C}{H_e+C}+\delta_p\right),\\ &= -\frac{\delta_p\beta_e\phi S^*_{pp}C}{(H_e+C^*)(H_e+C)}-\frac{\delta_p\beta_v\xi S^*_{pp}I_p}{(H_p+I_p)(H_p+I^*_{pp})}\\ &\quad -\left(\frac{\beta_v\xi S^*_{pp}}{H_p+I^*_{pp}}+\frac{\beta_e\phi S^*_{pp}}{H_e+C^*}\right)\left(\frac{\beta_v I_p}{H_p+I_p}+\frac{\beta_e C}{H_e+C}\right)<0.\end{aligned}$$

Moreover,

$$H^{-1} = \frac{1}{\det H}\begin{bmatrix} h_{11} & h_{12} & -\dfrac{(\delta_p+\mu_p)H_e\beta_e S^*_{pp}}{(H_e+C^*)(H_e+C)} \\ h_{21} & h_{22} & -\dfrac{\delta_p H_e\beta_e S^*_{pp}}{(H_e+C^*)(H_e+C)} \\ \dfrac{\beta_v\phi I_p}{H_p+I_p}+\dfrac{\beta_e\phi C}{H_e+C} & \delta_p\phi+\dfrac{\beta_v\phi I_p}{H_p+I_p}+\dfrac{\beta_e\phi C}{H_e+C} & h_{33} \end{bmatrix},$$

where,

$$\begin{aligned}
h_{11} &= \xi(\delta_p+\mu_p) - \frac{\beta_v\xi S^*_{pp}}{(H_p+I_p)(H_p+I^*_{pp})} - \frac{H_e\beta_e\phi S^*_{pp}}{(H_e+C^*)(H_e+C)},\\
h_{12} &= -\frac{\beta_v H_p\xi S^*_{pp}}{(H_p+I_p)(H_p+I^*_{pp})} - \frac{H_e\beta_e\phi S^*_{pp}}{(H_e+C^*)(H_e+C)},\\
h_{21} &= \frac{\beta_v\xi I_p}{H_p+I_p}+\frac{\beta_e\xi C}{H_e+C},\\
h_{22} &= \delta_p\xi+\frac{\beta_v\xi I_p}{H_p+I_p}+\frac{\beta_e\xi C}{H_e+C},\\
h_{33} &= (\delta_p+\mu_p)\left(\frac{\beta_v\xi I_p}{H_p+I_p}+\frac{\beta_e C}{H_e+C}\right)+\delta_p(\delta_p+\mu_p)-\frac{\delta_p\beta_v H_p S^*_{pp}}{(H_p+I_p)(H_p+I^*_{pp})}.
\end{aligned}$$

Using the fact that $\det H < 0$, and the relations that link the endemic equilibrium component, one can readily verify the hypotheses of Theorem 9 for the matrix $(H^{-1})^*$ and conclude that it is Volterra-Lyapunov stable. Hence, a 2×2 positive diagonal matrix $W^* = \text{diag}(\omega_1, \omega_2)$ exists such that $W^*(H^{-1})^* + (W^*(H^{-1})^*)^T < 0$. Setting $O = (-H)^{-1}$, we have $W^*O^* + (W^*O^*)^T > 0$. After lenghty but direct calculations, we obtain

$$(-\det H)[W^*O^* + (W^*O^*)^T] = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix},$$

with

$$a_{11} = 2\omega_1 h_{11},\ a_{12} = \omega_2 h_{21} - \omega_1 h_{12},\ a_{22} = 2\omega_2 h_{22}.$$

On the other hand,

$$W^*(-H)^* + (W^*(-H)^*)^T = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}, \quad \text{with}$$

$$
\begin{aligned}
b_{11} &= 2\omega_1\left(\delta_p + \frac{\beta_v I_p}{H_p + I_p} + \frac{\beta_e C}{H_e + C}\right),\\
b_{12} &= \omega_1 \frac{\beta_v H_p S^*_{pp}}{(H_p + I_p)(H_p + I^*_{pp})} - \omega_2\left(\frac{\beta_v I_p}{H_p + I_p} + \frac{\beta_e C}{H_e + C}\right),\\
b_{22} &= 2\omega_2\left(\delta_p + \mu_p - \frac{\beta_v H_p S^*_{pp}}{(H_p + I_p)(H_p + I^*_{pp})}\right).
\end{aligned}
$$

Next, we prove that $W^*(-H)^* + (W^*(-H)^*)^T > 0$. Indeed, since $W^*O^* + (W^*O^*)^T$ is positive definite and $-\det H > 0$, we have $\det\left\{(-\det H)[W^*O^* + (W^*O^*)^T]\right\} > 0$ and

$$
\begin{aligned}
\det\left\{(-\det H)[W^*O^* + (W^*O^*)^T]\right\} &= \xi^2 \det\left\{W^*(-H)^* + (W^*(-H)^*)^T\right\}\\
&\quad -4\omega_1\omega_2 \frac{\delta_p \xi H_e \beta_e \phi S^*_{pp}}{(H_e + C)(H_e + C^*)} - \omega_1^2\left[\frac{H_e \beta_e \phi S^*_{pp}}{(H_e + C)(H_e + C^*)}\right]^2\\
&\quad -2\omega_1\omega_2 \frac{H_e \xi \beta_e \phi S^*_{pp}}{(H_e + C)(H_e + C^*)}\left[\frac{\beta_v I_p}{H_p + I_p} + \frac{\beta_e C}{H_e + C}\right]\\
&\quad -2\omega_1^2 \frac{H_p H_e \beta_v \xi \beta_e \phi {S^*_{pp}}^2}{(H_p + I_p)(H_p + I^*_{pp})(H_e + C)(H_e + C^*)},
\end{aligned}
$$

Hence, the matrix $W^*(-H)^* + (W^*(-H)^*)^T$ is positive define. $W^*(-H)^* + (W^*(-H)^*)^T > 0$ and $W^*(-H^{-1})^* + (W^*(-H^{-1})^*)^T > 0$, then thanks to Theorem 10 (ii), there exists $\omega_3 > 0$ such that $W(-H) + (W(-H))^T > 0$; that is, $WH + H^TW < 0$. Thus H is Volterra-Lyapunov stable. Hence a feasible equilibrium x^* is globally asymptotically stable in Ω_1. ■

Next, we investigate the dynamics of the human sub-system:

$$
\frac{dy}{dt} = g(x^*; y) \equiv \begin{cases}
\dfrac{dS_h}{dt} = \Lambda_h + aE_h + \gamma I_h - (1 - cq)\tau_p \dfrac{S_h I^*_{pp}}{H_{ph} + I^*_{pp}} - (1 - cq)\tau_e \dfrac{S_h C^*}{H_{eh} + C^*} - \delta_h S_h,\\
\dfrac{dE_h}{dt} = (1 - cq)\tau_p \dfrac{S_h I^*_{pp}}{H_{ph} + I^*_{pp}} + (1 - cq)\tau_e \dfrac{S_h C^*}{H_{eh} + C^*} - (a + \delta_h + \epsilon)E_h,\\
\dfrac{dI_h}{dt} = \epsilon E_h - (\gamma + \mu_h + \delta_h)I_h.
\end{cases} \tag{48}
$$

Theorem 12 *The unique positive endemic equilibrium point $y^* = (S^*_h, E^*_h, I^*_h)$ of the system (48) is globally asymptotically stable if $\mathcal{R}_0^{pp} > 1$.*

Proof. We introduce the fractions $x = \delta_h S_h/\Lambda_h, y = \delta_h E_h/\Lambda_h, z = \delta_h I_h/\Lambda_h$ and scale time

by introducing a new time $\tau = \delta_h t$. This gives us the simplified system as:

$$\begin{cases} \dfrac{dx}{d\tau} = 1 + \overline{a}y + \overline{\gamma}z - (1-cq)\left[\overline{\tau}_p \dfrac{I^*_{pp}}{H_p + I^*_{pp}} + \overline{\tau}_e \dfrac{C^*}{H_e + C^*}\right] x - x, \\ \dfrac{dy}{d\tau} = (1-cq)\left[\overline{\tau}_p \dfrac{I^*_{pp}}{H_p + I^*_{pp}} + \overline{\tau}_e \dfrac{C^*}{H_e + C^*}\right] x - (1 + \overline{a} + \overline{\epsilon})y, \\ \dfrac{dz}{d\tau} = \overline{\epsilon}y - (1 + \overline{\mu}_h + \overline{\gamma})z, \end{cases} \tag{49}$$

where

$$\overline{a} = \frac{a}{\delta_h}, \overline{\tau}_p = \frac{\tau_p}{\delta_h}, \overline{\tau}_e = \frac{\tau_e}{\delta_h}, \overline{\epsilon} = \frac{\epsilon}{\delta_h}, \overline{\mu}_h = \frac{\mu_h}{\delta_h}, \overline{\gamma} = \frac{\gamma}{\delta_h},\ \overline{N} = x + y + z.$$

Here, we have used the fact that

$$\frac{d\overline{N}}{d\tau} = 1 - \overline{N} - \overline{\mu}_h z.$$

It can be shown that the region

$$\Omega_2 = \left\{(x, y, z) \in \mathbb{R}^3_+ / 0 \leq x + y + z \leq 1\right\},$$

is positively invariant. Now, consider the equivalent system:

$$\begin{cases} \dfrac{dy}{d\tau} = (1-cq)\left[\overline{\tau}_p \dfrac{I^*_{pp}}{H_p + I^*_{pp}} + \overline{\tau}_e \dfrac{C^*}{H_e + C^*}\right] x - (1 + \overline{a} + \overline{\epsilon})y, \\ \dfrac{dz}{d\tau} = \overline{\epsilon}y - (1 + \overline{\mu}_h + \overline{\gamma})z, \\ \dfrac{d\overline{N}}{d\tau} = 1 - \overline{N} - \overline{\mu}_h z. \end{cases} \tag{50}$$

Denote

$$\Omega_3 = \left\{(y, z, \overline{N}) \in \Omega_2 / \overline{N} = 1 - \overline{\mu}_h z\right\} = \left\{(y, z, \overline{N}) \in \Omega_2 / x + y + (1 + \overline{\mu}_h)z = 1\right\}.$$

Then it not difficult to prove that Ω_3 is a positively invariant and attracting subset of Ω_2. Next we use the Poincaré-Bendixson techniques to prove that system (48) has no periodic solution. Let us assume that the system (48) has a periodic solution $\psi(\tau) = \{x(\tau), y(\tau), z(\tau)\}$. Let $\psi(\tau)$ be the trajectory of periodic solution, and Π be the planar region of $\psi(\tau)$. Let $f_1(x, y, z), f_2(x, y, z)$ and $f_3(x, y, z)$ respectively represent the three expressions of the right-hand side of the system (49). Set $\mathbf{f} = (f_1, f_2, f_3)^T$, $\mathbf{g}(x, y, z) = \mathbf{r} \times \mathbf{f}/(xyz)$, where $\mathbf{r} = (x, y, x)^T$. Then $\mathbf{g} \cdot \mathbf{f} = 0$, let $\mathbf{g} = (g_1, g_2, g_3)$, where

$$g_1 = \frac{f_3(x, z)}{xz} - \frac{f_2(x, y)}{xy}\ ,\ g_2 = \frac{f_1(x, y)}{xy} - \frac{f_3(y, z)}{yz}\ ,\ g_3 = \frac{f_2(y, z)}{yz} - \frac{f_1(x, z)}{xz}.$$

Then

$$Curl\mathbf{g} = \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}\ ,\quad \frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\ ,\quad \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right).$$

By simple calculations, we have

$$\begin{aligned}
\frac{f_1(x,y)}{xy} &= \frac{1}{xy}+\frac{\overline{a}}{x}+\frac{\overline{\gamma}(1-x-y)}{(1+\overline{\mu}_h)xy}-\frac{(1-cq)\left[\overline{\tau}_p\dfrac{I^*_{pp}}{H_p+I^*_{pp}}+\overline{\tau}_e\dfrac{C^*}{H_e+C^*}\right]}{y}-\frac{1}{y},\\
\frac{f_1(x,z)}{xz} &= \frac{1}{xz}+\frac{\overline{a}[1-x-(1+\overline{\mu}_h)z]}{xz}+\frac{\overline{\gamma}}{x}-\frac{(1-cq)\left[\overline{\tau}_p\dfrac{I^*_{pp}}{H_p+I^*_{pp}}+\overline{\tau}_e\dfrac{C^*}{H_e+C^*}\right]}{z}-\frac{1}{z},\\
\frac{f_2(y,z)}{yz} &= \frac{(1-cq)\left[\overline{\tau}_p\dfrac{I^*_{pp}}{H_p+I^*_{pp}}+\overline{\tau}_e\dfrac{C^*}{H_e+C^*}\right][1-y-(1+\overline{\mu}_h)z]}{yz}-\frac{1+\overline{a}+\overline{\epsilon}}{z},\\
\frac{f_2(x,y)}{xy} &= \frac{(1-cq)\left[\overline{\tau}_p\dfrac{I^*_{pp}}{H_p+I^*_{pp}}+\overline{\tau}_e\dfrac{C^*}{H_e+C^*}\right]}{y}-\frac{1+\overline{a}+\overline{\epsilon}}{x},\\
\frac{f_3(y,z)}{yz} &= \frac{\overline{\epsilon}}{z}-\frac{1+\overline{\gamma}+\overline{\mu}_h}{y},\\
\frac{f_3(x,z)}{xz} &= \frac{\overline{\epsilon}[1-x-(1+\overline{\mu}_h)z]}{xz}-\frac{1+\overline{\gamma}+\overline{\mu}_h}{x}.
\end{aligned}$$

Now, since $x+y+(1+\overline{\mu}_h)z=1$, it is clear that $-[1-(1+\overline{\mu}_h)z]=-(x+y)<0$, so that

$$\frac{\partial g_3}{\partial y}-\frac{\partial g_2}{\partial z}=-\frac{\overline{\epsilon}}{z^2}-\frac{(1-cq)\left[\overline{\tau}_p\dfrac{I^*_{pp}}{H_p+I^*_{pp}}+\overline{\tau}_e\dfrac{C^*}{H_e+C^*}\right][1-(1+\overline{\mu}_h)z]}{y^2z}<0.$$

Further, we have

$$\begin{aligned}
\frac{\partial g_1}{\partial z}-\frac{\partial g_3}{\partial x} &= -\frac{\overline{\epsilon}(1-x)}{xz^2}-\frac{1}{x^2z}-\frac{\overline{a}[1-(1+\overline{\mu}_h)z]}{x^2z}-\frac{\overline{\gamma}}{x^2},\\
\frac{\partial g_2}{\partial x}-\frac{\partial g_1}{\partial y} &= -\frac{1}{x^2y}-\frac{\overline{a}}{x^2}-\frac{\overline{\gamma}(1-y)}{(1+\overline{\mu}_h)x^2y}-\frac{(1-cq)\left[\overline{\tau}_p\dfrac{I^*_{pp}}{H_p+I^*_{pp}}+\overline{\tau}_e\dfrac{C^*}{H_e+C^*}\right]}{y^2}.
\end{aligned}$$

Obviously, the right hand sides in the two expressions above are negative. Taking the unit normal vector of Ω_3

$$\mathbf{n}=\frac{(1,1,1+\overline{\mu}_h)^T}{\sqrt{\overline{\mu}_h^2+2\overline{\mu}_h+3}},$$

we obtain $(Curl\mathbf{g})\cdot\mathbf{n}<0$. By the Poincaré-Bendixson theorem, we know that the system (48) has no periodic solution. Thus, the equilibrium y^* is GAS in Ω_2.

Finally, the combination of Theorem 8, Theorem 11 and Theorem 12 establishes the GAS of E^*_{pp} ■

References

[1] N. H. Gao, Z. H. Lu, B. Cao et al, "*Clinical findings in 111 cases of influenza A (H7N9) virus infection,*" *the New England Journal of Medicine,* vol. 368, no. 24, pp. 2277-2285, 2013.

[2] T. T.-Y. Lam, J. Wang, and Y. Shen, "*The genesis and source of the H7N9 influenza viruses causing human infections in China,* Nature," vol. 502, no. 7470, pp. 241-244, 2013.

[3] WHO, *Number of confirmed human cases of avian influenza A (H7N9) reported to World Health Organization, World Health Organization, 2013,* https://www.who.int/influenza/human-animal-interface/influenza-h7n9/WHO-H7N9-review-31May13.pdf

[4] Centers for Disease Control and Prevention (2010) *Key facts about avian influenza (bird flu) and highly pathogenic avian influenza A (H7N9) virus.* http://www.cdc.gov/flu/avian/gen-info/facts. htm (Accessed 28 Jan 2013).

[5] WHO, *Frequently Asked Questions on human infection caused by the avian influenza A(H7N9) virus.* https://www.who.int/influenza/human-animal-interface/faq-H7N9/en/ (Accessed 14 February 2014).

[6] Food and Agriculture Organization of the United Nations (2004) *FAO recommendations on the prevention, control and eradication of highly pathogenic avian influenza (HPAI) in Asia (proposed with the support of the OIE)*. http://www.fao.org/ag/againfo/programmes/en/programmes.htm (Accessed 30 Jan 2013).

[7] Food and Agriculture Organization of the United Nations (2008) *Possible modes of transmission of avian viruses to people: studies in experimental models.* http://www.fao.org/docs/eims/upload//250678/aj155e00.pdf (Accessed 30 Jan 2013)

[8] R. Gao, B. Cao, and Y. Hu, "*Human infection with a novel avianorigin influenza A (H7N9) virus,*" the New England Journal of Medicine, vol. 368, no. 20, pp. 1888-1897, 2013.

[9] T. Watanabe, M. Kiso, S. Fukuyama et al., "*Characterization of H7N9 influenza A viruses isolated from humans,*" Nature, vol.501, no. 7468, pp. 551-555, 2013.

[10] C.-J. Bao, L.-B. Cui, M.-H. Zhou, L. Hong, G. F. Gao, and H. Wang, "*Live-animal markets and influenza A (H7N9) virus infection,*" the New England Journal of Medicine, vol. 368, no. 24, pp. 2337-2339, 2013.

[11] D. M. Rao, A. Chernyakhovsky, and V. Rao, "*Modeling and analysis of global epidemiology of avian influenza,*" Environmental Modeling and Sofware, vol. 24, no. 1, pp. 124-134, 2009.

[12] M.-T. Li, Z. Jin, G.-Q. Sun, and J. Zhang, "*Modeling direct and indirect disease transmission using multi-group model,*" Journal of Mathematical Analysis and Applications, vol. 446, no. 2, pp. 1292-1309, 2017.

[13] Y. Xiao, X. Sun, S. Tang, and J. Wu, "*Transmission potential of the novel avian influenza A(H7N9) infection in mainland China,*" Journal of theoretical Biology, vol. 352, pp. 1-5, 2014.

[14] Nunõ M, Chowell G, Gumel AB (2006) *Assessing the role of basic control measures, antivirals and vaccine in curtailing pandemic influenza:* scenarios for the US, UK and the Netherlands. J R Soc Interface 4:505-521.

[15] Sanhong Liu, Shigui Ruan, Xinan Zhang, *On avian influenza epidemic models with time delay.* Theory Biosci (2015).DOI 10.1007/s12064-015-0212-8.

[16] Xinhong Zhang, *Global dynamics of a stochastic avian-human influenza epidemic model with logistic growth for avian population.* Nonlinear Dyn (2017) 90:2331-2343. DOI 10.1007/s11071-017-3806-5

[17] Xinan Zhang, Lan Zou, Jing Chen, Yile Fang, Jicai Huang, Jinhui Zhang, Sanhong Liu, Guangting Feng, Cuihong Yang and Shigui Ruan, *Avian influenza A (H7N9) virus has been established in CHINA.* Journal of Biological Systems, Vol. 25, No. 4 (2017) 605-623. World Scientific Publishing Company, DOI: 10.1142/S0218339017400095.

[18] Arnaud Feukouo Fossi, Jean Lubuma, Calvin Tadmon, Berge Tsanou : *Mathematical modeling and nonstandard finite difference scheme analysis for the environmental and spillover transmission of an Avian Influenza Virus model.* Dynamical Systems, An International Journal, soumis 13 mars 2020.

[19] Food and Agriculture Organization of the United Nations (2004) *FAO recommendations on the prevention, control and eradication of highly pathogenic avian influenza (HPAI) in Asia (proposed with the support of the OIE).* http://www.fao.org/ag/againfo/ programmes/en/programmes.htm (Accessed 30 Jan 2013).

[20] Food and Agriculture Organization of the United Nations (2008) *Possible modes of transmission of avian viruses to people: studies in experimental models.* http://www.fao.org/docs/ eims/upload//250678/aj155e00.pdf (Accessed 30 Jan 2013).

[21] Marangon S, Cecchinato M, Capua I (2008) *Use of vaccination in avian influenza control and eradication.* Zoonoses Public Health 55:65-72.

[22] A. Berman, R.J. Plemmons, *Nonnegative matrices in the mathematical sciences,* SIAM., 1994.

[23] Peiris, J.S.M., De Jong, M.D., Guan, Y, *Avian influenza virus (H5N1) : a threat to human health.* Clin. Microbiol. Rev. 20:2 (2007) 243-267.

[24] Killingley .B, Nguyen-Van-Tam .J, *Routes of influenza transmission.* Influenza Other Respir. Viruses 7(2013) 42-51.

[25] Dent, J.E, Kao R.R, Kiss .I.Z, Hyder .K, Arnold .M, *Contact structures in the poultry industry in Great Britain: exploring transmission routes for a potential avian influenza virus epidemic.* BMC Vet. Res 4:27 (2008) 1746-6148.

[26] Shu-Min Guo, Juan Wang, Mini Ghost and Xue-Zhi, *Analysis of Avian Influenza* A $(H7N9)$ *model based on the low pathogenicity in poultry,* J.Biol.Syst, 25:2 (2017) 1-16.

[27] Iwami S, Takeuchi Y, Korobeinikov A, Liu X, *Prevention of avian influenza epidemic: What policy should we choose?* J.Theor.Biol 252 (2008) 732-741.

[28] Iwami S, Takeuchi Y, Liu X, *Avian-human influenza epidemic model,* Math.Biosci 207 (2007) 1-25.

[29] Iwami S, Takeuchi Y, Liu X, *Avian flu pandemic: Can we prevent it?* J.Theor.Biol 257 (2009) 181-190.

[30] Bourouiba L, Teslya A, Wu J, *Highly pathogenic avian influenza outbreak mitigated by seasonal low pathogenic strains: Insights from dynamic modeling,* J.Theor.Biol 271 (2011) 181-201.

[31] Chong KC, Wang X, Liu S, Cai J, Su X, Zee BC, Tam G, Wang MH, Chen E, *Interpreting the transmissibility of the avian influenza A(H7N9) infection from 2013 to 2015 in Zhejiang Province, China,* Epidemiol.Infect 144, (2016) 1584-1591.

[32] Liu Z, Fang C-T, *A modeling study of human infections with avian influenza A H7N9 virus in mainland China,* Int.J.Infect.Dis,41 (2015) 73-78.

[33] Chen Y, Wen Y, *Global dynamic analysis of a H7N9 avian-human influenza model in an outbreak region,* J.Theor.Biol , 367 (2015) 180-188 .

[34] Liu S, Pang L, Ruan S, Zhang X, *Global dynamics of avian influenza epidemic models with psychological effect,* Comput.Math.Methods.Med, Article ID 913726,(2015).

[35] Redheffer R. *Volterra multipliers ii.* SIAM J.Alg.Disc.Math; (1985). 6, 612–623.

[36] P. van den Driessche and J. Watmough, *Reproduction numbers and sub-threshold endemic equilibria for compartmental models of desease transmission,* Math.Biosci 180 (2002) 29-48.

[37] J. P. LaSalle, *The Stability of Dynamical Systems,* Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, (1976).

[38] Castillo-Chavez et al, *On the computation of Ro and its role on global stability* , (2001).

[39] N.S Chong, J.M Tchuenche, R.J. Smith: *A mathematical model of avian influenza with half-saturated incidence.* Theory Biosci. (2014) 133:23-38 DOI 10.1007/s12064-013-0183-6.

[40] H. Lee, A. Lao: *Transmission dynamics and control strategies assessment of avian influenza (H5N6) in the Philippines.* infectious disease modelling (2018) 35-59.

[41] C. Castillo-Chavez and B. Song, *Dynamical model of tuberclosis and their applications,* Math.Biosci.Eng, 1 (2004), 361-404.

[42] Martin Luther Mann Manyombe, Joseph Mbang, Jean Lubuma and Berge Tsanou, *Global dynamics of a vaccination model for infectious deseases with asymptomatic carriers.* Math.Biosci.Eng. Volume 13, Number 4,(2016), pp. 813-840.

[43] J. Gjorgjieva, K. Smith, G. Chowell, F. Sanchez, J. Snyder and C. Castillo-Chavez, *The role of vaccination in the control of SARS,* Math.Biosci.Eng., 2 (2005), 753-769.

[44] Cross GW, *Three types of matrix stability.* Linear Algebra Appl 20:(1978) , 253-263.

[45] Goh BS, *Global stability in two species interactions.* J.Math.Biol 3:(1976), 313-318.

[46] M. Vidyasagar, *Decomposition techniques for large-scale systems with nonadditive interactions : Stability and stabilizability.* IEEE Trans. Autom.Control, 25(1980) :773-779.

[47] J. Carr, *Applications Centre Manifold theory.* Springer-Verlag, New York, 1981.

[48] Berge Tsanou, Samuel Bowong, Jean Lubuma, Joseph Mbang, *Assessing the impact of the environmental contamination on the transmission of Ebola virus disease (EVD).* J.Appl.Math.Comput. DOI 10.1007/s12190-016-1033-8.

[49] G. Djatcha Yaleu, S. Bowong, E. Houpa Danga, J. Kurths, *Mathematical analysis of the dynamical transmission of Neisseria meningitidis serogroup A.* Int.J.Comput.Math, DOI: 10.1080/00207160.(2017).1283411.

[50] Y. Song, T. Zhang, *Global stability of the positive equilibrium of a mathematical model for unstirred membrane reactors,* Bull. Korean Math. Soc. 54 (2017) 383-389.

[51] S. Segal and A. Pollard, *Vaccines against bacterial meningitis,* Br. Med. Bull. 72 (2004), pp. 65-81. Available at http://bmb.oxfordjournals.org.

[52] A. Scherer and A. McLean, *Mathematical models of vaccination,* Br. Med. Bull. 62 (2002), pp. 187-192.

[53] H.S. Rodrigues, M.T. Monteiro, and D.F.M. Torres, *Vaccination models and optimal control strategies to dengue,* Math. Biosci. 247 (2014), pp. 1-12.

[54] S Marino, IB Hogue, CJ Ray, DE Kirschner, *A methodology for performing global uncertainty and sensitivity analysis in systems biology,* J. theor. biol, 254 (2008): 178-196.

[55] Boloye Gomero, *Latin Hypercube Sampling and Partial Rank Correlation Coefficient Analysis Applied to an Optimal Control Problem,* Master's Thesis, University of Tennessee, 2012.

[56] S. Gandon, M. Mackinnon, S. Nee, and A. Read, *Imperfect Vaccination: Some Epidemiological and Evolutionary Consequences,* in: Proceedings of Biological Sciences, 2003, pp. 1129-1134.

[57] C. Farrington, *On vaccine efficacy and reproduction numbers,* Math. Biosci. 185 (2003), pp. 89-101.

[58] C. Bowman, A.B. Gumelc, P. van den Driessched, J. Wue, H. Zhue: *A mathematical model for assessing control strategies against West Nile virus.* Bulletin of Mathematical Biology 67 (2005) 1107-1133.

[59] J-C. Kamgang and G. Sallet, *Computation of threshold conditions for epidemiological models and global stability of the disease-free equilibrium (DFE),* Math. Biosci. 213 (2008), 1–12.

Table 1: Description of the variables and associated parameters.

Symbols	Description
S_p	Susceptible poultry
V_p	Vacinated poultry
I_p	Infected poultry
S_h	Susceptible human
E_h	Latent human
I_h	Infected human
C	Concentration of virus
Λ_p	Numbers of imported poultry
β_v	Direct contact rate in poultry host
β_e	Indirect contact rate in poultry host
δ_p	Natural death rate of poultry
c	Percentage of the population employing personal protection
q	Efficiency of personal protection
H_p	Half-saturation constant for poultry with AI virus
H_e	Half-saturation constant for aerosol with AI virus
H_{ph}	Half-saturation constant for humans with AI virus contracted from infected poultry
H_{eh}	Half-saturation constant for humans with AI virus contracted from infected aerosol
Λ_h	Recruitment rate of humans
a	Recovery rate of the latent humans
γ	Recovery rate of the infected humans
μ_h	Disease-related death rate for humans
τ_p	Rate at which poultry-to-human avian influenza is contracted
ϵ	Morbidity of the latent humans
δ_h	Natural death rate of humans
ξ	Degradation rate of virus
τ_e	Rate at which environment-to-human avian influenza is contracted
ϕ	Emission rate of virus by poultry
π	Prevalence rate of the vaccination program
ν	Vaccine efficacy
μ_p	AI induced poultry mortality

Table 2: Sensitivity indexes for $\mathcal{R}_0$.

Parameter	Sensitivity index	Value	Parameter	Sensitivity index	Value
β_v	S_{β_v}	$8.7605 \cdot 10^3$	π	S_π	$-1.2929 \cdot 10^{26}$
β_e	S_{β_e}	$1.1356 \cdot 10^{16}$	ν	S_ν	$-1.2929 \cdot 10^{26}$
Λ_p	S_{Λ_p}	1	μ_p	S_{μ_p}	$-7.9319 \cdot 10^{23}$
ϕ	S_ϕ	$1.1356 \cdot 10^{16}$	H_p	S_{H_p}	$-8.7605 \cdot 10^3$
ξ	S_ξ	$-1.1356 \cdot 10^{16}$	H_e	S_{H_e}	$-1.1356 \cdot 10^{16}$
δ_p	S_{δ_p}	$-2.4940 \cdot 10^{25}$			

Table 3: Parameter ranges for PRCC analysis.

Parameters	values	Range variation at 10%	Range variation at 50%
Λ_p	10000	[9000 – 11000]	[5000 – 15000]
Λ_h	15	[13.5 – 16.5]	[7.5 – 22.5]
π	0.40	[0.36 – 0.44]	[0.2 – 0.6]
δ_p	0.01389	[0.0125 – 0.0153]	[6.95$\cdot 10^{-3}$ – 0.021]
δ_h	0.00025641	[2.31$\cdot 10^{-4}$ – 2.82$\cdot 10^{-4}$]	[1.28$\cdot 10^{-4}$ – 3.85$\cdot 10^{-4}$]
H_{ph}	120000	[108000 – 132000]	[60000 – 180000]
H_{eh}	10000	[9000 – 11000]	[5000 – 15000]
ν	0.65	[0.585 – 0.715]	[0.325 – 0.975]
β_e	0.002	[1.8$\cdot 10^{-3}$ – 2.2$\cdot 10^{-3}$]	[0.001 – 0.003]
β_v	1.7143$\cdot 10^{-6}$	[1.54$\cdot 10^{-6}$ – 1.89$\cdot 10^{-6}$]	[8.57$\cdot 10^{-7}$ – 2.57$\cdot 10^{-6}$]
H_p	180000	[162000 – 198000]	[90000 – 270000]
H_e	10^6	[9$\cdot 10^5$ – 11$\cdot 10^5$]	[5$\cdot 10^5$ – 15$\cdot 10^5$]
μ_p	0.04	[0.036 – 0.044]	[0.02 – 0.06]
μ_h	0.001	[9$\cdot 10^{-4}$ – 1.1$\cdot 10^{-3}$]	[5$\cdot 10^{-4}$ – 1.5$\cdot 10^{-4}$]
ϕ	10^5	[9000 – 11000]	[50000 – 150000]
τ_p	0.6	[0.54 – 0.66]	[0.3 – 0.9]
τ_e	0.1	[0.09 – 0.11]	[0.05 – 0.15]
a	1	[0.9 – 1.1]	[0.5 – 1.5]
ξ	2000	[1800 – 2200]	[1000 – 3000]
γ	0.9	[0.81 – 0.99]	[0.45 – 1.35]
ϵ	1	[0.9 – 1.1]	[0.5 – 1.5]
$dummy$	1	—	—

Table 4: PRCC of model's parameters (Range variation at 10%).

	PRCCs and significance						
Parameters	S_p	V_p	I_p	C	S_h	E_h	I_h
Λ_p	0.9568**	0.9571**	0.8613**	-0.7837**	0.8244**	0.8196**	0.8875**
Λ_h	-0.0044	0.0486	-0.0180	0.7633**	0.1658**	0.1998**	-0.0218
π	-0.8910**	0.9615**	-0.4592**	0.3496**	-0.4179**	-0.4165**	-0.5023 **
δ_p	-0.9529**	-0.9558**	-0.9012**	0.8257**	-0.8604**	-0.8650**	-0.9166**
δ_h	-0.0598	-0.0148	0.0185	-0.3984**	-0.0902*	-0.0496	-0.0231
H_{ph}	-0.0115	-0.0270	-0.0144	0.0427	-0.0107	-0.0316	-0.0647
H_{eh}	-0.0085	-0.0108	0.0543	0.0433	-0.1076*	-0.0177	0.0288
ν	0.1276**	0.0237	-0.4559**	0.3827**	-0.4175**	-0.3480**	-0.5226**
β_e	-0.3385**	-0.1178**	0.8576**	-0.7673**	0.8125**	0.8223**	0.8863**
β_v	0.0227	-0.0055	-0.0260	-0.0110	0.0374	0.0134	-0.0019
H_p	0.0076	0.0166	-0.0439	-0.0411	0.0654	-0.0427	-0.0005
H_e	0.3070**	0.0545	-0.8603**	0.7699**	-0.8147**	-0.8178**	-0.8841**
μ_p	0.2575**	0.1233*	-0.7881**	0.6818**	-0.7344**	-0.7183**	-0.8233**
μ_h	-0.0058	-0.0163	-0.0018	-0.0424	-0.0673	0.0353	0.0425
ϕ	-0.3268**	-0.1088**	0.8508**	-0.7777**	0.8164**	0.8229**	0.8882**
τ_p	-0.0067	-0.0058	0.0367	0.0057	0.0525	0.0157	0.0471
τ_e	0.0050	0.0057	0.0002	-0.1978**	0.2247**	0.2747**	-0.0260
a	0.0328	0.0086	0.0392	0.1334**	-0.0668	-0.1878 **	0.0318
ξ	0.2856**	0.1339*	-0.8589**	0.7799**	-0.8177**	-0.8170**	-0.8867**
γ	0.0543	-0.0080	0.0330	0.1876**	0.0361	-0.3127**	-0.0287
ϵ	0.0466	-0.0344	0.0289	-0.1201*	-0.1723**	0.1859**	-0.0155
$dummy$	0.0320	0.0054	-0.0106	-0.0352	-0.0313	0.0018	0.0222

*: p-value < 0.01, **: p-value < 0.001.

Table 5: PRCC of model's parameters (Range variation at 50%).

	PRCCs and significance						
Parameters	S_p	V_p	I_p	C	S_h	E_h	I_h
Λ_p	0.9676**	0.9561**	0.8629**	-0.6300**	0.7096**	0.6490**	0.8576**
Λ_h	0.0112	-0.0034	0.0055	0.8874**	0.3523**	0.3370**	0.0303
π	-0.8536**	0.9595**	-0.2348**	0.1304**	-0.1714**	-0.1900**	-0.1955**
δ_p	-0.9628**	-0.9535 **	-0.9104**	0.7137**	-0.7810**	-0.7433**	-0.9026**
δ_h	-0.0207	0.0311	0.0549	-0.5294**	-0.1849**	-0.1252*	0.0321
H_{ph}	0.0097	0.0215	0.0324	0.1981**	-0.2619**	-0.1561**	-0.0082
H_{eh}	0.0342	0.0096	0.0186	0.0081	-0.0044	0.0012	0.0296
ν	0.0011	0.1363**	-0.3133**	0.1204**	-0.1759**	-0.1218 **	-0.2785**
β_e	-0.3523**	-0.1684**	0.8575**	-0.6362**	0.6991**	0.6293**	0.8479**
β_v	-0.0018	-0.0116	0.0435	-0.0212	-0.0612	0.0282	0.0319
H_p	0.0139	-0.0314	0.0111	0.0002	0.0419	-0.0106	0.0140
H_e	0.2435**	0.1114**	-0.7613**	0.5405**	-0.5698**	-0.5366**	-0.7468**
μ_p	0.1671**	0.0774	-0.7944**	0.4905**	-0.6226**	-0.5699**	-0.7962**
μ_h	0.0035	-0.0128	0.0199	-0.1273**	-0.0325	-0.0366	-0.0114
ϕ	-0.2273**	-0.10777*	0.7568**	-0.5662**	0.6296**	0.5802**	0.8578**
τ_p	0.0024	0.0493	-0.0240	-0.2690**	0.3096**	0.2361**	-0.0020
τ_e	0.0230	-0.0311	0.0173	-0.3918**	0.4711**	0.3429**	-0.0214
a	-0.0269	-0.0686	-0.0030	0.3049**	-0.4066**	-0.3143**	-0.0189
ξ	0.2755**	0.0840	-0.7641**	0.5393**	-0.5999**	-0.5688**	-0.8588**
γ	-0.0068	0.0036	0.0049	0.3469**	0.1410*	-0.5831**	0.0072
ϵ	-0.0042	-0.0536	-0.0027	-0.1090**	-0.4621**	0.3347**	0.0444
$dummy$	0.0021	-0.0190	-0.0262	-0.0342	0.0693	-0.0139	0.0141

**: p-value < 0.001, *: p-value < 0.01.

Table 6: The eight most influential parameters of model system (3).

	Number of state variables significantly correlate		
Parameters	Range 10%	Range 50%	Total
Λ_p	7	7	14
π	3	2	5
δ_p	7	7	14
β_e	5	5	10
H_e	5	5	10
μ_p	5	5	10
ϕ	5	5	10
ξ	5	5	10

Table 7: Parameters and baseline values.

Symbols	Estimate for AIV	Source	Symbols	Estimate for AIV	Source
Λ_p	10000 ind	Assumed	ξ	2000 week^{-1}	Assumed
β_v	$1.71 \cdot 10^{-6} week^{-1}$	[34]	τ_e	0.1 week^{-1}	Assumed
β_e	0.002 week^{-1}	Assumed	ϕ	$10^5 g \cdot m^3 \cdot ind^{-1} \cdot week^{-1}$	Assumed
δ_p	1/72 week^{-1}	[32]	π	0.40 week^{-1}	Assumed
H_p	180000 ind	[39]	c	0.9	Assumed
Λ_h	15 ind	Assumed	q	0.9	Assumed
a	1 week^{-1}	[34]	ν	0.65	Assumed
γ	0.9 week^{-1}	[34]	H_{ph}	120000 ind	[39]
μ_h	0.001 week^{-1}	[26]	H_{eh}	10000 ind	Assumed
τ_p	0.6 week^{-1}	[26]	δ_h	0.00025641 week^{-1}	[32]
ϵ	1 week^{-1}	[26]	H_e	$10^6 g.m^3$	[26]
μ_p	1/25 week^{-1}	Assumed			