A Finite Model Property for Gödel Modal Logics

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1 Introduction

Gödel modal logics combine Kripke frames of modal logics with the semantics of the well-known fuzzy (and intermediate) Gödel logic. These logics, in particular, analogues GK (for "fuzzy" frames) and $\mathsf{GK^C}$ (for "crisp" frames) of the modal logic K, have been investigated in some detail by Caicedo and Rodríguez [5, 4] and Metcalfe and Olivetti [11, 12]. More general approaches, focussing mainly on finite-valued modal logics, have been developed by Fitting [7, 8], Priest [13], and Bou et al. [2]. Multimodal variants of GK have also been proposed as the basis for fuzzy description logics in [10] and (restricting to finite models) [1].

Axiomatizations were obtained for the box and diamond fragments of GK (where the box fragments of GK and $\mathsf{GK^C}$ coincide) in [5] and for the diamond fragment of $\mathsf{GK^C}$ in [12]. It was subsequently shown in [4] that the full logic GK is axiomatized either by adding the Fischer Servi axioms for intuitionistic modal logic IK (see [6]) to the union of the axioms for both fragments, or by adding the prelinearity axiom for Gödel logic to IK. Decidability of the diamond fragment of GK was established in [5], using the fact that the fragment has the finite model property with respect to its Kripke semantics. This finite model property fails for the box fragment of GK and $\mathsf{GK^C}$ and the diamond fragment of $\mathsf{GK^C}$, but decidability and PSPACE-completeness for these fragments was established in [11, 12] using analytic Gentzen-style proof systems.

The first main contribution of the work reported here is a decidability proof for validity in full GK and $\mathsf{GK^C}$ that makes use of alternative Kripke semantics for these logics admitting the finite model property. The key idea of this new semantics is to restrict evaluations of modal formulas at a world to a particular finite set of truth values. A similar strategy is used to establish decidability, and indeed co-NP-completeness, for the crisp Gödel modal logic $\mathsf{GS5^C}$ based on $\mathsf{S5}$ frames where accessibility is an equivalence relation. Moreover, this logic, an extension of the intuitionistic modal logic MIPC of Bull [3] and Prior [14] with prelinearity and a further modal axiom, corresponds exactly to the one-variable fragment of first-order Gödel logic (see [9]).[1]

2 Gödel Modal Logics

Gödel modal logics are defined based on a language $\mathcal{L}_{\Box\Diamond}$ consisting of a fixed countably infinite set Var of (propositional) variables, denoted $p, q, \ldots$, binary connectives $\to$, $\land$, $\lor$, constants $\bot$, $\top$, and unary operators $\Box$ and $\Diamond$. The set of *formulas* $\mathrm{Fml}_{\Box\Diamond}$, with arbitrary members denoted $\varphi, \psi, \ldots$ is defined inductively as usual.

[1] A full paper with the same title as this extended abstract will be presented at WoLLIC 2013 and may be downloaded from www.philosophie.ch/297.

We also fix the *length* of a formula φ, denoted $\ell(\varphi)$, to be the number of symbols occurring in φ and define $\neg\varphi = \varphi \to \bot$.

The standard semantics of Gödel logic is characterized by the Gödel t-norm min and its residuum $\to_{\mathsf{G}}$, defined on the real unit interval $[0,1]$ by

$$x \to_{\mathsf{G}} y = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise.} \end{cases}$$

The Gödel modal logics GK and $\mathsf{GK^C}$ are defined semantically as generalizations of the modal logic K where connectives behave at a given world as in Gödel logic.

A *fuzzy Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ where W is a non-empty set of *worlds* and $R\colon W \times W \to [0,1]$ is a binary *fuzzy accessibility relation* on W. If $Rxy \in \{0,1\}$ for all $x, y \in W$, then R is called *crisp* and $\mathfrak{F}$, a *crisp Kripke frame*. In this case, we often write $R \subseteq W \times W$ and Rxy to mean $Rxy = 1$.

A GK-*model* is a triple $\mathfrak{M} = \langle W, R, V\rangle$, where $\langle W, R\rangle$ is a fuzzy Kripke frame and $V : \mathrm{Var} \times \mathrm{W} \to [0,1]$ is a mapping, called a *valuation*, extended to $V : \mathrm{Fml}_{\Box\Diamond} \times \mathrm{W} \to [0,1]$ as follows:

$$\begin{array}{rcl} V(\bot, x) & = & 0 \\ V(\top, x) & = & 1 \\ V(\varphi \to \psi, x) & = & V(\varphi, x) \to_{\mathsf{G}} V(\psi, x) \\ V(\varphi \land \psi, x) & = & \min(V(\varphi, x), V(\psi, x)) \\ V(\varphi \lor \psi, x) & = & \max(V(\varphi, x), V(\psi, x)) \\ V(\Box\varphi, x) & = & \inf\{Rxy \to_{\mathsf{G}} V(\varphi, y) : y \in W\} \\ V(\Diamond\varphi, x) & = & \sup\{\min(Rxy, V(\varphi, y)) : y \in W\}. \end{array}$$

A $\mathsf{GK^C}$-*model* satisfies the extra condition that $\langle W, R\rangle$ is a crisp Kripke frame. In this case, the conditions for $\Box$ and $\Diamond$ may also be read as

$$\begin{array}{rcl} V(\Box\varphi, x) & = & \inf(\{1\} \cup \{V(\varphi, y) : Rxy\}) \\ V(\Diamond\varphi, x) & = & \sup(\{0\} \cup \{V(\varphi, y) : Rxy\}). \end{array}$$

A formula $\varphi \in \mathrm{Fml}_{\Box\Diamond}$ is *valid* in a GK-model $\mathfrak{M} = \langle W, R, V\rangle$ if $V(\varphi, x) = 1$ for all $x \in W$. If φ is valid in all L-models for some logic L (in particular GK or $\mathsf{GK^C}$), then φ is said to be L-*valid*, written $\models_{\mathsf{L}} \varphi$.

Let us agree to call a model *finite* if its set of worlds is finite, and say that a logic has the *finite model property* if validity in the logic coincides with validity in all finite models of the logic. In [5], it is shown that the formula $\Box\neg\neg p \to \neg\neg\Box p$ is valid in all finite GK models, but not in the infinite crisp model $\langle \mathbb{N}, R, V\rangle$ where $Rxy = 1$ for all $x, y \in \mathbb{N}$ and $V(p, x) = 1/(x+1)$ for all $x \in \mathbb{N}$. That is, neither GK nor $\mathsf{GK^C}$ has the finite model property.

3 A New Semantics and Finite Model Property

In order for a $\mathsf{GK^C}$-model to render $\varphi = \Box\neg\neg p \to \neg\neg\Box p$ invalid at a world x, there must be values of p at worlds accessible to x that form an infinite descending sequence tending to but never reaching 0. This ensures that the infinite model falsifies φ, but also that no particular world acts as a "witness" to the value of $\Box p$. Our strategy is to redefine models to allow only a finite number of values at each world that can be taken by box-formulas and diamond-formulas. A formula such as $\Box p$ can then be "witnessed" at a world where the value of p is merely "sufficiently close" to the value of $\Box p$.

Let us define a GFK-*model* as a quadruple $\mathfrak{M} = \langle W, R, T, V\rangle$, where $\langle W, R, V\rangle$ is a GK-model and $T \colon W \to \mathcal{P}_{<\omega}([0,1])$ is a function from worlds to finite sets of truth values satisfying $\{0,1\} \subseteq T(x) \subseteq [0,1]$ for all $x \in W$. If $\langle W, R, V\rangle$ is also a GK^C-model, then $\mathfrak{M}$ will be called a GFK^C-*model*.

The GFK-valuation V is extended to formulas using the same clauses for non-modal connectives as for GK-valuations, together with the revised modal connective clauses:

$$\begin{array}{rcl} V(\Box\varphi, x) & = & \max\{r \in T(x) : r \leq \inf\{Rxy \to_\mathsf{G} V(\varphi, y) : y \in W\}\} \\ V(\Diamond\varphi, x) & = & \min\{r \in T(x) : r \geq \sup\{\min(Rxy, V(\varphi, y)) : y \in W\}\}. \end{array}$$

As before, a formula $\varphi \in \mathrm{Fml}_{\Box\Diamond}$ is *valid* in a GFK-model $\mathfrak{M} = \langle W, R, T, V\rangle$ if $V(\varphi, x) = 1$ for all $x \in W$, written $\mathfrak{M} \models_\mathsf{GFK} \varphi$.

Observe now that for the formula $\Box\neg\neg p \to \neg\neg\Box p$, there are very simple finite GFK^C-counter-models: for example, $\mathfrak{M}_0 = \langle W, R, T, V\rangle$ with $W = \{a\}$, $Raa = 1$, $T(a) = \{0,1\}$, and $V(p,a) = \frac{1}{2}$. It is easy to see that $V(\neg p, a) = 0$, $Raa \to_\mathsf{G} V(\neg\neg p, a) = 1$, and so $V(\Box\neg\neg p, a) = 1$. Moreover, $V(\Box p, a) = 0$ (since $Raa \to_\mathsf{G} V(p,a) = \frac{1}{2}$, and 0 is the next smaller element of $T(a)$); hence $V(\neg\Box p, a) = 1$ and $V(\neg\neg\Box p, a) = 0$. So $1 = V(\Box\neg\neg p, a) > V(\neg\neg\Box p, a) = 0$ and $\mathfrak{M}_0 \not\models_{\mathsf{GFK}^\mathsf{C}} \Box\neg\neg p \to \neg\neg\Box p$.

Indeed, it can be shown that every formula φ that is not GFK-valid (or GFK^C-valid) has a finite GFK (respectively, GFK^C) counter-model of size exponential in the length of φ. It follows that validity in GFK and GFK^C is decidable. Moreover, since validity in GK and GK^C can be shown to correspond exactly to validity in GFK and GFK^C, respectively, decidability follows also for these logics. More precisely, we have established:

Theorem 1. *For each* $\varphi \in \mathrm{Fml}_{\Box\Diamond}$*:*

(a) $\models_\mathsf{GK} \varphi$ *iff* $\models_\mathsf{GFK} \varphi$ *iff* φ *is valid in all* GFK*-models* $\mathfrak{M} = \langle W, R, T, V\rangle$ *satisfying* $|W| \leq (\ell(\varphi)+2)^{\ell(\varphi)}$ *and* $|T(x)| \leq \ell(\varphi)+2$ *for all* $x \in W$.

(b) $\models_{\mathsf{GK}^\mathsf{C}} \varphi$ *iff* $\models_{\mathsf{GFK}^\mathsf{C}} \varphi$ *iff* φ *is valid in all* GFK^C*-models* $\mathfrak{M} = \langle W, R, T, V\rangle$ *satisfying* $|W| \leq (\ell(\varphi)+2)^{\ell(\varphi)}$ *and* $|T(x)| \leq \ell(\varphi)+2$ *for all* $x \in W$.

Moreover, validity in GK *and* GK^C *is decidable.*

4 A Crisp Gödel S5 Logic

The crisp Gödel modal logic $\mathsf{GS5}^\mathsf{C}$ is characterized by validity in GK^C-models where R is an equivalence relation. This logic may also be viewed as the one-variable fragment of first-order Gödel logic (see [9]).

We define a $\mathsf{GFS5}^\mathsf{C}$-*model* as a GFK^C-model $\mathfrak{M} = \langle W, R, T, V\rangle$ such that $\langle W, R, V\rangle$ is a $\mathsf{GS5}^\mathsf{C}$-model and also $T(x) = T(y)$ whenever Rxy (ensuring that formulas of the form $\Box\varphi$ and $\Diamond\varphi$ receive the same truth value in all worlds of the same equivalence class). We are then able to show that every non-$\mathsf{GFS5}^\mathsf{C}$-valid formula φ has a finite $\mathsf{GFS5}^\mathsf{C}$-counter-model of size linear in the length of φ. Since again we are able to establish a correspondence between $\mathsf{GFS5}^\mathsf{C}$-validity and $\mathsf{GS5}^\mathsf{C}$-validity, we obtain the following:

Theorem 2. *For each* $\varphi \in \mathrm{Fml}_{\Box\Diamond}$*:* $\models_{\mathsf{GS5^C}} \varphi$ *iff* $\models_{\mathsf{GFS5^C}} \varphi$ *iff* φ *is valid in all* $\mathsf{GFS5^C}$*-models* $\mathfrak{M} = \langle W, R, T, V\rangle$ *where* $|W| \leq \ell(\varphi) + 2$ *and* $|T(x)| \leq \ell(\varphi) + 2$ *for all* $x \in W$*. Moreover, validity in* $\mathsf{GS5^C}$ *and the one-variable fragment of first-order Gödel logic is decidable and indeed co-NP-complete.*

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