



Tight Complexity for Modal Extensions of Lec with K, 4, T

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Abstract

Substructural logics are defined by dropping some of the structural properties of intuitionistic logic such as commutativity (exchange), contraction, or weakening. Substructural logics with contraction and without weakening can have very high complexity: specifically, the Full Lambek calculus with exchange and contraction (FLec) is complete for the Ackermannian complexity class. Here, ‘Full’ refers to the presence of additive (lattice) connectives; they are crucial for simulating zero-tests in the lower bounding argument. Indeed, Schmitz (2016) showed that its multiplicative fragment **Lec** has much lower complexity: provability and deducibility are 2EXPTIME-complete. In this paper, we investigate the effect of adding the standard modal rules K, 4, and T to this multiplicative fragment.

We show that provability in **Lec**+K and **Lec**+4 remains in 2EXPTIME by extending Schmitz’s expansive BVASS argument. We also obtain Ackermannian lower bounds for **Lec**+K deducibility, by constructing a reduction from an expansive counter machine. In the latter, the K rule is used to simulate zero-tests. Some of these results are robust under the addition of the T rule. When combined with known results, this yields a tight classification: **Lec**+K (+T) provability and **Lec**+4 provability are 2EXPTIME-complete, and **Lec**+K (+T) deducibility is Ackermannian-complete.

1 Introduction

Substructural logics arise by omitting structural rules such as exchange (e), weakening (w), contraction (c), or associativity from classical or intuitionistic proof calculi.

$$\frac{\Gamma, \alpha, \gamma, \Delta \Rightarrow \beta}{\Gamma, \gamma, \alpha, \Delta \Rightarrow \beta} (e) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma, \alpha \Rightarrow \beta} (w) \qquad \frac{\Gamma, \alpha, \alpha \Rightarrow \beta}{\Gamma, \alpha \Rightarrow \beta} (c)$$

The absence of structural rules means that assumptions can no longer be freely permuted, duplicated, or discarded. This yields a family of systems that contains linear logic, relevance logics, mathematical fuzzy logics, and the Lambek calculus, with applications ranging from programming-language semantics, type theory, separation logic, program verification, and resource-sensitive computation, to reasoning under uncertainty and linguistics. Moreover, the proof calculi that are so obtained typically exhibit substantially higher computational complexity than the familiar classical or intuitionistic propositional systems.

This phenomenon is especially visible in (propositional) Full Lambek calculi with contraction and no weakening. Urquhart [19] showed that **FLec** (essentially intuitionistic propositional calculus minus weakening) is ACK-complete, and Chvalovský and Horčík [4] established undecidability of its non-commutative variant **FLc**. Here, ACK is the Ackermannian complexity class

in Schmitz’s [15] fast-growing complexity hierarchy. Informally, this is the class of problems decidable by a Turing machine in time (equivalently, space) of a single Ackermannian function closed under primitive recursive functions. Simply put, a problem that is ACK-hard cannot be solved by any algorithm that runs in primitive recursive time/space in the input length.

Urquhart obtained the lower bound for FLec by a reduction from expansive additive branching counter machines. The additive (lattice) connective $\vee$ in the logical language plays a crucial role: it simulates the additive branching rule of the counter machine, which in turn is equivalent to a zero-test on the counters. This raised a natural question: what is the complexity of the multiplicative fragment Lec of FLec i.e., in the absence of the additive connectives?

Schmitz [16] answered this question by establishing 2EXPTIME-completeness of Lec provability. The lower bound was obtained by a reduction from branching vector addition system with states (BVASS) coverability. The upper bound adapts the branching VASS coverability argument of Demri et al [5]—itself an extension of Rackoff’s [13] classic EXPSPACE coverability argument for VAS—to accommodate contraction (note that coverability admits any increase whereas only non-zero values may strictly increase under contraction). Lazić and Schmitz [10] later obtained the upper bound also via an algebraic analysis of well-quasi-order bad sequences.

Schmitz only considered provability in Lec because his result readily extends to deducibility via the following *semi-deduction theorem*: if $\Gamma \Rightarrow \beta$ is deducible in Lec from a finite set $\{\Gamma_i \Rightarrow \beta_i\}_{i \in T}$ of assumptions, then there is some subset $X \subseteq \{\odot \Gamma_i \rightarrow \beta_i\}_{i \in T}$ such that $X, \Gamma \Rightarrow \beta$ is provable, where $\odot \Gamma$ be the fusion of all members in the multiset Γ (under some enumeration), where fusion $\cdot$ is multiplicative conjunction. The idea is to replace every assumption leaf $\Gamma_i \Rightarrow \beta_i$ with the provable $\odot \Gamma_i \rightarrow \beta_i, \Gamma_i \Rightarrow \beta_i$ and proceed downwards as before to the root, then use contraction to remove duplicates. It follows that the Lec deducibility problem can be decided in 2EXPTIME, simply by checking provability of $X, \Gamma \Rightarrow \beta$ for each of the exponentially-many options for X .

Here we study the complexity of provability and deducibility of extensions of Lec by the standard modal rules, after identifying which cut-elimination/restriction properties hold.

$$\frac{\Gamma \Rightarrow \beta}{\Box \Gamma \Rightarrow \Box \beta} \text{ K} \qquad \frac{\Box \Gamma, \Gamma \Rightarrow \beta}{\Box \Gamma \Rightarrow \Box \beta} \text{ 4} \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma, \Box \alpha \Rightarrow \beta} \text{ T}$$

This question lies at the intersection of three lines of research. First, it contributes to the rich literature on substructural modal logics, as fragments of linear logic and even more generally (see e.g., [6, 11, 9, 18]). Secondly, it complements recent work (see e.g., [7, 14, 2]) on the computability and complexity of axiomatic—as opposed to language—extensions of base systems. Thirdly, it fits into the growing exchange of methods between infinite-state systems and proof theory (see e.g., [19, 8, 1]). A natural starting point to answer this question would be to try and extend Schmitz’s arguments for Lec to the present setting. However, we face the following obstacles (in several cases, their significance is genuine and reflected in the complexity results):

Firstly, the semi-deduction theorem for Lec described above does not always hold in this setting, and provability and deducibility then become distinct problems. As an example of where the semi-deduction theorem argument breaks down, observe that there is no obvious bound on the number of applications of the modal rule K in a Lec+K deduction; we can still convert an assumption into an axiom as before, but the additional formula $\odot \Gamma_i \rightarrow \beta_i$ that we added will be prefixed by a box each time it passes from the premise to the conclusion of a K rule. As a consequence, the number of candidates for the set X becomes too large. Indeed, we show here that Lec+K provability is 2EXPTIME-complete but deducibility is ACK-complete.

Secondly, every formula in the conclusion $\Box \Gamma \Rightarrow \Box \beta$ of the K and 4 rules must be boxed, and every leading box must be removed in the premise. It is a challenge to implement this in the BVASS that Schmitz employs in his upper bounding argument for Lec provability (owing to

Calculus	Provability (lb/ub)	Deducibility (lb/ub)
Lec+K	2EXPTIME [16] / 2EXPTIME [here]	ACK [here] / ACK [2]
Lec+K+T	2EXPTIME [16] / 2EXPTIME [here]	ACK [here] / ACK [2]
Lec+4	2EXPTIME [16] / 2EXPTIME [here]	2EXPTIME [16] / ACK [2]
Lec+4+T	2EXPTIME [16] / ACK [2]	2EXPTIME [16] / ACK [2]
Lec+T	2EXPTIME [16] / 2EXPTIME [here]	2EXPTIME [16] / 2EXPTIME [here]

Table 1: Summary of complexity results in the vicinity of logics under consideration.

the absence of zero-test capability in BVASS). We get around this issue, for Lec+K provability and Lec+4 provability, by exploiting the existence of a bound on modal depth in a proof (unlike in a deduction), and assigning distinct counters in the BVASS for each modal depth.

Thirdly, although the simulation of zero-tests using lattice connectives already appears in the undecidability of linear logic (Lincoln et al [12]), simulation of zero-tests using the modal rules instead (for deducibility in Lec+K and Lec+K+T) requires a new encoding.

This contributions of this paper are summarised in Table 1.

2 Preliminaries

Let $\mathcal{P}$ be a countably infinite set of *propositional variables*. The set of *formulas* is defined in the usual way using the grammar $\alpha := p \in \mathcal{P} \mid \mathbf{1} \mid \mathbf{0} \mid \alpha \cdot \alpha \mid \alpha \rightarrow \alpha \mid \Box \alpha \mid \epsilon$.

We usually denote the empty formula ϵ (i.e., the absence of a formula) with a blank.

A sequent has the form $\Gamma \Rightarrow \beta$ where Γ (‘antecedent’) is a multiset of formulas and β (‘succedent’) is a formula. The sequent calculus Lec is presented in Figure 1. A *deduction* of sequent s in Lec from a finite set $\mathcal{T}$ of sequents (‘theory’) is a finite tree obeying the rules of Lec such that the root is labelled s , and every leaf is an instance of an axiom or from $\mathcal{T}$. If $\mathcal{T} = \emptyset$, then we call it a proof, and we say that s is provable.

$$\begin{array}{c}
\frac{}{\alpha \Rightarrow \alpha} \quad \frac{}{\mathbf{0} \Rightarrow} \quad \frac{}{\Rightarrow \mathbf{1}} \quad \frac{\Gamma \Rightarrow \beta}{\mathbf{1}, \Gamma \Rightarrow \beta} \mathbf{1} \text{ L} \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathbf{0}} \mathbf{0} \text{ R} \\
\\
\frac{\Gamma, \alpha, \beta \Rightarrow \gamma}{\Gamma, \alpha \cdot \beta \Rightarrow \gamma} (\cdot \text{L}) \quad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta} (\cdot \text{R}) \quad \frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta \Rightarrow \gamma}{\Gamma, \Delta, \alpha \rightarrow \beta \Rightarrow \gamma} (\rightarrow \text{L}) \\
\\
\frac{\Gamma, \alpha, \alpha \Rightarrow \beta}{\Gamma, \alpha \Rightarrow \beta} (\text{c}) \quad \frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta} (\text{cut}) \quad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} (\rightarrow \text{R})
\end{array}$$

Figure 1: The sequent calculus Lec.

Let $\text{sub}(s)$ denote the set of subformulas in the sequent s . The modal depth $\text{md}(\alpha)$ of a formula α is the maximum nesting of boxes in α i.e., $\text{md}(\Box \alpha) = \text{md}(\alpha) + 1$, and $\text{md}(\alpha) = 0$ if α does not contain any boxes. The modal depth $\text{md}(s)$ of sequent s is maximum of $\text{md}(\alpha)$ over every formula α occurring in the antecedent or succedent of s .

Although the results in Ramanayake [14] and Balasubramanian et al [2] concern axiomatic—rather than modal language—extensions of FLec, the following is an easy extension of those arguments.

Theorem 1. *Deducibility in Lec+K (+T) and Lec+4 (+T) are in ACK.*

The idea is to consider backward proof search restricted to branches that are controlled bad sequences with respect to the well-quasi-ordering induced by contraction. The complexity bound is then an immediate consequence of the maximum length of such controlled bad sequences.

3 Cut-elimination and Standard Deductions

The lower bounding is achieved by establishing a reduction from a counter machine acceptance to provability/deducibility. Specifically, this means (i) simulating a computation in a proof/deduction, and (ii) extracting a computation from any proof/deduction that corresponds to a computation statement. Informally speaking, a proof/deduction is more liberal than a computation, hence it is helpful to restrict our consideration to well-behaved proofs and deductions i.e., cut-free proofs and standard deductions, defined below.

We observe that cut-elimination for $\text{Lec}+4+\text{T}$ might be deduced from cut-elimination for linear logic since the $!$ exponential behaves like the S4 modality (see also [3]). In any case, the results we need are stated below.

Lemma 2. *A proof in $\text{Lec}+\text{K}$ ($+\text{T}$) with cuts can be transformed into a proof without cuts of the same sequent. The same holds for $\text{Lec}+\text{T}$ and $\text{Lec}+4$ ($+\text{T}$).*

We observe in passing that $\text{Lec}+\text{K}+4$ and $\text{Lec}+\text{K}+4+\text{T}$ do not have cut-elimination. Indeed, consider the following provable sequent in $\text{Lec}+\text{K}+4$. A case analysis on the non-cut rules applicable to $\Box p, \Box p \Rightarrow \Box(p \cdot \Box p \cdot p)$ reveals that this sequent does not have a cut-free proof.

$$\frac{\frac{p, \Box p \Rightarrow p \cdot \Box p}{\Box p \Rightarrow \Box(p \cdot \Box p)} 4 \quad \frac{p \cdot \Box p, p \Rightarrow p \cdot \Box p \cdot p}{\Box(p \cdot \Box p), \Box p \Rightarrow \Box(p \cdot \Box p \cdot p)} \text{K}}{\Box p, \Box p \Rightarrow \Box(p \cdot \Box p \cdot p)} \text{cut}$$

Standard deductions

Following Urquhart [19], a sequent $\Gamma \Rightarrow \beta$ is *regular* if Γ consists of propositional variables. A *regular theory* is a finite set of regular sequents. A deduction from a regular theory $\mathcal{T}$ such that every cut has as its left premise a leaf from $\mathcal{T}$ ('standard cut') is called a *standard deduction*.

Lemma 3. *Let $\mathcal{T}$ be a regular theory. Any deduction of s in $\text{Lec}+\text{K}+\text{T}$ from $\mathcal{T}$ can be transformed into a standard deduction of s from $\mathcal{T}$, such that every formula in this standard deduction is a subformula of $\mathcal{T} \cup \{s\}$.*

4 $\text{Lec}+\text{K}$ provability is 2EXPTIME (upper bound)

We present a polytime reduction from provability to acceptance in an expansive branching vector addition system with states. In the latter, expansive refers to the fact that non-zero counters in this machine can freely increase in value during the computation.

Definition 4. *An expansive branching vector addition system (EBVASS, for short) is a tuple (k, Q, R_1, R_2, C_m) where $k \in \mathbb{N}$ is the number of dimensions, Q is a finite set of states, $R_1 \subseteq Q \times \mathbb{Z}^k \times Q$ is a finite set of unary rules including an expansive rule for each coordinate (defined below), $R_2 \subseteq Q \times Q \times Q$ is a finite set of binary rules, and $C_m \subseteq Q \times \mathbb{N}^k$ is a finite set of BVASS axioms.*

The operational semantics of EBVASS are rules operating on configurations of the form $(q, \mathbf{c}) \in Q \times \mathbb{N}^k$. Applying a rule to a configuration yields a configuration, or a pair of configurations. The available rules are

- $(q, \mathbf{c}) \xrightarrow{r_1} (q', \mathbf{c} + \mathbf{c}')$ for some rule $r_1 = (q, \mathbf{c}', q') \in R_1$
- $(q, \mathbf{c}) \xrightarrow{r_2} (q', \mathbf{c}'), (q'', \mathbf{c}'')$ for some rule $r_2 = (q, q', q'') \in R_2$ and $\mathbf{c} = \mathbf{c}' + \mathbf{c}''$
- $(q, \mathbf{c}) \xrightarrow{e} (q, \mathbf{c}')$ if $c_i \leq c'_i$, and $c_i=0$ implies $c'_i=0$ for all i . This is called an *expansive* rule.

An EBVASS B *accepts* a configuration $C = (q, \mathbf{c})$ if there exists a tree ('reachability witness') rooted at C obeying $\xrightarrow{R_1}, \xrightarrow{R_2}, \xrightarrow{e}$ such that every node is labelled by a configuration and every leaf is an EBVASS axiom, i.e., in C_m . In a configuration, the vector is in $\mathbb{N}^k$, so it cannot be negative in any coordinate. The following EBVASS acceptance problem (a special case of the reachability problem) is in 2EXPTIME [5, 16]:

Problem 1. *Given an EBVASS B and configuration $(q, \mathbf{c})$, does B accept $(q, \mathbf{c})$?*

Note that the input for the formal problem is an encoding $\langle B, (q, \mathbf{c}) \rangle$ of $(B, (q, \mathbf{c}))$.

Suppose that s is provable in $\text{Lec} + \text{K}$. By cut-elimination, lemma 2, there is a cut-free proof of s . Note that the number of K rules on any branch of π is bounded by the modal depth $d = \text{md}(\Gamma \Rightarrow \beta)$, since the K strictly decreases the modal depth by one from conclusion to premise. In the following, we motivate the definition of a EBVASS B_s , and the extraction of a reachability witness that is obtained by tracing upwards in the cut-free proof from the root towards the leaves. Specifically, in B_s , the state q_β^i will be used to denote a sequent in the proof with succedent β that has seen i applications of the K rule (counting from the root).

Now let us discuss how the vector in the configuration is used to represent the antecedent of a sequent in the proof, given a sequent $\Gamma \Rightarrow \beta$, we will reserve one coordinate for each subformula of the sequent, and the succedent is represented using the state of the EBVASS. A rule like $\cdot L$ with principal formula $\alpha \cdot \beta$ can be simulated from conclusion to premise, following Schmitz, by decrementing a coordinate corresponding to $\alpha \cdot \beta$ and incrementing the coordinates corresponding to α and β . However, something more is required to handle the K rule: after all, every boxed formula in the conclusion loses its leading box in the premise, so we need to ensure that the coordinate of every such boxed formula becomes 0. If this is not ensured, the EBVASS might accept a configuration corresponding to a sequent, even though the sequent itself is not provable. This could be ensured with a zero-test, but that is not an option here, as reachability in an EBVASS with zero-test is ACK-hard.

Our solution is to use multiple coordinates for each antecedent subformula that correspond to the number of K rules from the root in the proof. Specifically, add the unary rule $(q_{\square\beta}^i, 0, q_{K,\beta}^{i+1})$ that takes us to a new auxiliary state $q_{K,\beta}^{i+1}$. Also, add an auxiliary unary rule $(q_{K,\beta}^{i+1}, -e_{\square\alpha}^i + e_\alpha^{i+1}, q_{K,\beta}^{i+1})$ which stays in this auxiliary state and converts $e_{\square\alpha}^i$ to e_α^{i+1} (see this as the implementation of $\square\alpha$ in the conclusion becoming α in the premise). Finally, add one further auxiliary unary rule $(q_{K,\beta}^{i+1}, 0, q_\beta^{i+1})$ that takes us out from the auxiliary state to q_β^{i+1} , so that the system is ready to simulate the next proof rule. The point is that if any $e_{\square\alpha}^i$ is not decremented before moving out of the auxiliary state $q_{K,\beta}^{i+1}$, then that coordinate will have no further opportunity to decrement. This means that it will persist for the rest of the computation, making it impossible to reach an accepting configuration (q_α^j, e_α^j) (notice that the superscript is the same in both terms in the latter).

Formal definition. For sequent $s = (\Gamma \Rightarrow \beta)$ with modal depth d , define EBVASS B_s as:

- Allocate a coordinate c_α^i for each $i \leq d$ and $\alpha \in \text{sub}(s)$. Hence $k = (d+1)|\text{sub}(s)|$. Define e_α^i to be the c_α^i unit vector i.e., $(0, \dots, 0, 1, 0, \dots, 0)$ where the solitary 1 is at coordinate c_α^i .
- Define a state q_β^i for $i \leq d$ and $\beta \in \text{sub}(s)$. Further states are needed to implement the left implication rule: $\{q_\beta^{i, \alpha_1 \rightarrow 1 \alpha_2} \mid \alpha_1 \rightarrow \alpha_2, \beta \in \text{sub}(s), i \leq d\}$ and $\{q_\beta^{i, \alpha \rightarrow 2 \alpha_2} \mid \alpha_1 \rightarrow \alpha_2, \beta \in \text{sub}(s), i \leq d\}$, and the following to implement the K rule: $\{q_{K, \beta}^i \mid \beta \in \text{sub}(s), i \leq d\}$
- $\rightarrow R$ is implemented via the unary rules $\{(q_{\alpha \rightarrow \beta}^i, e_\alpha^i, q_\beta^i) \mid \alpha \rightarrow \beta \in \text{sub}(s), i \leq d\}$
- $\rightarrow L$ is implemented via the unary rules

$$\begin{aligned} & \{(q_\beta^i, -e_{(\alpha_1 \rightarrow \alpha_2)}^i, q_\beta^{i, \alpha_1 \rightarrow 1 \alpha_2}) \mid \alpha_1 \rightarrow \alpha_2, \beta \in \text{sub}(s), i \leq d\} \\ & \{(q_\beta^{i, \alpha_1 \rightarrow 2 \alpha_2}, e_{\alpha_2}^i, q_\beta^i) \mid \alpha_1 \rightarrow \alpha_2, \beta \in \text{sub}(s), i \leq d\} \end{aligned}$$

and the binary rules

$$\{(q_\beta^{i, \alpha_1 \rightarrow 1 \alpha_2}, q_{\alpha_1}^i, q_\beta^{i, \alpha_1 \rightarrow 2 \alpha_2}) \mid \alpha_1 \rightarrow \alpha_2, \beta \in \text{sub}(s), i \leq d\}$$

- $\cdot L$ is implemented via $\{(q_\beta^i, -e_{(\alpha_1 \cdot \alpha_2)}^i + e_{\alpha_1}^i + e_{\alpha_2}^i, q_\beta^i) \mid \alpha_1 \cdot \alpha_2, \beta \in \text{sub}(s), i \leq d\}$
- If $\mathbf{1} \in \text{sub}(s)$, we implement $\mathbf{1}L$ via: $\{q_\beta^i, -e_{\mathbf{1}}^i, q_\beta^i \mid \beta \in \text{sub}(s), i \leq d\}$
- If $\mathbf{0} \in \text{sub}(s)$, we implement $\mathbf{0}R$ via: $\{q_0^i, 0, q^i \mid i \leq d\}$
- $\cdot R$ is implemented via the binary rules $\{(q_{\beta_1 \cdot \beta_2}^i, q_{\beta_1}^i, q_{\beta_2}^i) \mid \beta_1 \cdot \beta_2 \in \text{sub}(s), i \leq d\}$
- The K rule is implemented via the unary rules

$$\begin{aligned} & \{(q_{\square \beta}^i, 0, q_{K, \beta}^{i+1}) \mid \square \beta \in \text{sub}(s), i \leq d\} \\ & \{(q_{K, \beta}^{i+1}, -e_{\square \alpha}^i + e_\alpha^{i+1}, q_{K, \beta}^{i+1}) \mid \square \alpha, \square \beta \in \text{sub}(s), i \leq d\} \\ & \{(q_{K, \beta}^{i+1}, 0, q_\beta^{i+1}) \mid \square \beta \in \text{sub}(s), i \leq d\} \end{aligned}$$

- The set of axioms is $\{(q_\alpha^i, e_\alpha^i) \mid \alpha \in \text{sub}(s), i \leq d\}$. If $\mathbf{0} \in \text{sub}(s)$, we add (q^i, e_0^i) , and if $\mathbf{1} \in \text{sub}(s)$, we add $(q_1^i, 0)$.

For a sequent $\alpha_1, \dots, \alpha_k \Rightarrow \beta$, and $0 \leq i \leq d$, we define the configuration

$$\Theta^i(\alpha_1, \dots, \alpha_k \Rightarrow \beta) = (q_\beta^i, e_{\alpha_1}^i + \dots + e_{\alpha_k}^i) = (q_\beta^i, \mathbf{c})$$

4.1 From Lec+K provability to EBVASS acceptance

The following lemma extracts a reachability witness from a cut-free proof of t . This is straightforward since B_t was defined by anticipating the rule instances that can appear in the proof.

Lemma 5. *Given a cut-free proof π of $t = (\Gamma \Rightarrow \beta)$, then $\Theta^i(s)$ is an accepting configuration in B_t , where $s \in \pi$ and i is the number of K applications from the root of π to s .*

Proof sketch. By Theorem 2, every sequent t has a cut-free proof in $\text{Lec}+\text{K}$. We establish the claim by induction on the height of the sequent in the proof.

Base case: The height of s is 0, which can only happen when it is an axiom $\alpha \Rightarrow \alpha$. In this case, $\Theta^i(s) = (q_\alpha^i, e_\alpha^i)$, which is an axiom of B_t , and so it is reachable.

Inductive case: Case analysis on the rule instance in π that concludes in s . Only a couple cases are shown in this proof sketch.

- The rule instance concluding in s is left fusion:

$$\frac{\Gamma, \alpha, \alpha' \Rightarrow \beta}{\Gamma, \alpha \cdot \alpha' \Rightarrow \beta} \cdot L$$

By IH, the premise $\Theta^i(s')$ an accepting configuration. From $\Theta^i(s)$ we can reach this using the rule $r = (q_\beta^i, -e_{(\alpha_1 \cdot \alpha_2)}^i + e_{\alpha_1}^i + e_{\alpha_2}^i, q_\beta^i)$:

$$\begin{aligned} \Theta^i(s) &= (q_\beta^i, \mathbf{c} + k_1 e_{\alpha_1}^i + k_2 e_{\alpha_2}^i + (k_3 + 1) e_{(\alpha_1 \cdot \alpha_2)}^i + 1) \\ &\xrightarrow{r} (q_\beta^i, \mathbf{c} + (k_1 + 1) e_{\alpha_1}^i + (k_2 + 1) e_{\alpha_2}^i + k_3 e_{(\alpha_1 \cdot \alpha_2)}^i) = \Theta^i(s') \end{aligned}$$

So $\Theta^i(s)$ is also an accepting configuration.

- The rule instance concluding in s is the K rule:

$$\frac{\Gamma \Rightarrow \beta}{\Box \Gamma \Rightarrow \Box \beta} \text{K}$$

By IH, C_t accepts $\Theta^{i+1}(s') = (q_\beta^{i+1}, \mathbf{c}^{i+1})$, where the multiset $\Gamma = \alpha_1, \dots, \alpha_k$ is represented by the vector $\mathbf{c} = e_{\alpha_1} + \dots + e_{\alpha_k}$. Define $\Box \mathbf{c}^i := e_{\Box \alpha_1}^i + \dots + e_{\Box \alpha_k}^i$. Starting from $\Theta^i(s)$, our task is to reach $\Theta^{i+1}(s')$. First, apply the rule $r_1 = (q_{\Box \beta}^i, 0, q_{\mathbf{K}, \beta}^{i+1})$:

$$\Theta^i(s) = (q_{\Box \beta}^i, \Box \mathbf{c}^i) \xrightarrow{r_1} (q_{\mathbf{K}, \beta}^{i+1}, \Box \mathbf{c}^i)$$

Now, repeatedly apply $r_2 = (q_{\mathbf{K}, \beta}^{i+1}, -e_{\Box \alpha}^i + e_{\alpha}^{i+1}, q_{\mathbf{K}, \beta}^{i+1})$ to reach ultimately

$$(q_{\mathbf{K}, \beta}^{i+1}, \Box \mathbf{c}^i) \xrightarrow{r_2^*} (q_{\mathbf{K}, \beta}^{i+1}, \mathbf{c}^{i+1})$$

Finally, apply $r_3 = (q_{\mathbf{K}, \beta}^{i+1}, 0, q_\beta^{i+1})$ to reach

$$(q_{\mathbf{K}, \beta}^{i+1}, \mathbf{c}^{i+1}) \xrightarrow{r_3} (q_\beta^{i+1}, \mathbf{c}^{i+1}) = \Theta^{i+1}(s')$$

□

4.2 Reverse direction

We now transform a suitable reachability witness into a proof, taking care to show that the EBVASS does not exhibit behaviour that cannot be simulated by the proof rules.

Lemma 6. *Let s and t be sequents such that $\text{sub}(s) \subseteq \text{sub}(t)$. If B_t accepts $\Theta^i(s)$ for some $i \leq \text{md}(t)$, then s is provable.*

Proof sketch. Induction on the length of the reachability witness for $\Theta^i(s)$.

Base case: The length of the computation is 0, this can only accept configurations which are axioms: $(q_\alpha^i, 0, \dots, 1, \dots, 0)$, where the 1 occurs at the index of counter c_α^i , which represents the sequent $\alpha \Rightarrow \alpha$ which is an axiom, and so it is provable.

Inductive case: Case analysis on the EBVASS rule that was applied to get $\Theta^i(s') = (q_\beta^i, c_{\alpha_1}, \dots, c_{\alpha_k})$. We discuss selected cases.

- The rule used was an expansion rule, in this case we get:

$$\Theta^i(s) = (q_\beta^i, \mathbf{c} + (k_1 + 1)e_\alpha^i) \xrightarrow{e} (q_\beta^i, \mathbf{c} + (k_1 + 2)e_\alpha^i) = \Theta^i(s')$$

We can get:

$$\frac{\Gamma, \alpha, \alpha \Rightarrow \beta}{\Gamma, \alpha \Rightarrow \beta} c$$

By induction hypothesis, the top sequent $s' = \Gamma, \alpha, \alpha \Rightarrow \beta$ is provable.

- The rule was a rule of the form $(q_{\alpha \rightarrow \beta}^i, e_\alpha^i, q_\beta)$, in this case we get:

$$\Theta^i(s) = (q_{\alpha \rightarrow \beta}^i, \mathbf{c} + (k_1)e_\alpha^i) \rightarrow (q_\beta^i, \mathbf{c} + (k_1 + 1)e_\alpha^i) = \Theta^i(s')$$

We can get:

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} \rightarrow R$$

By induction hypothesis, the top sequent $s' = \Gamma, \alpha \Rightarrow \beta$ is provable.

- The last rule used was a rule of the form: $r = (q_{\square\beta}^i, 0, q_{\mathbf{K},\beta}^{i+1})$, in this case we get: The novel case is when the first rule is, so

$$\Theta^i(s) = (q_{\square\beta}^i, \mathbf{c}^i) \xrightarrow{r} (q_{\mathbf{K},\beta}^{i+1}, \mathbf{c}^i)$$

The only rules that are applicable in state $q_{\mathbf{K},\beta}^{i+1}$ are $r_1 = (q_{\mathbf{K},\beta}^{i+1}, -e_{\square\alpha}^i + e_{\square\alpha}^{i+1} + e_\alpha^{i+1}, q_{\mathbf{K},\beta}^{i+1})$, $r_2 = (q_{\mathbf{K},\beta}^{i+1}, 0, q_\beta^{i+1})$, and the expansive rules. The only way to reach an axiom—as we must, since the hypothesis is acceptance—is to apply r_2 at some point, as that is the only way to get out of state $q_{\mathbf{K},\beta}^{i+1}$. Crucially, if r_2 were applied when $\mathbf{c}^i$ is not empty, then the non-empty coordinates would remain positive for the rest of the run; we could never reach an axiom, contradicting the acceptance hypothesis. It follows that, immediately prior to the r_2 application, we must have reached $(q_{\mathbf{K},\beta}^{i+1}, \square\mathbf{d}^{i+1})$ such that $\mathbf{c} \leq \mathbf{d}$ and $c_j = 0$ implies $d_j = 0$, where equality holds iff no expansion rule was applied. Now we have

$$(q_{\mathbf{K},\beta}^{i+1}, \square\mathbf{d}^{i+1}) \xrightarrow{r} (q_\beta^{i+1}, \square\mathbf{d}^{i+1}) = \Theta^{i+1}(s')$$

By IH, s' is provable. Now obtain the proof of s as follows, where $\Theta^{i+1}(s') = \Gamma' \Rightarrow \beta$:

$$\frac{\frac{\Gamma' \Rightarrow \beta}{\Gamma \Rightarrow \beta} c^*}{\square\Gamma \Rightarrow \square\beta} \mathbf{K}$$

□

Theorem 7. *Lec+K provability is 2EXPTIME-complete*

Proof sketch. Let t be any sequent. By inspection, $\langle t \rangle \mapsto \langle B_t, \Theta^0(t) \rangle$ is polytime constructible. Suppose that t is provable. Then, there is a cut-free proof π of t . By Lemma 5, B_t accepts $\Theta^0(t)$. For the reverse direction, suppose that B_t accepts $\Theta^0(t)$. Lemma 6 tells us that t is provable. So we have demonstrated a polytime reduction from Lec+K provability to EBVASS acceptance. The latter is in 2EXPTIME [16] so the upper bound claim follows. For the lower bound: Schmitz [16] showed that Lec provability is 2EXPTIME-hard. The trivial reduction from the latter to Lec+K provability (note that cut-elimination in Lec+K implies that any Lec formula that is provable in Lec+K is provable in Lec) yields the desired lower bound is provable in Lec+K iff it is provable in Lec. □

5 Lec+4 provability is 2EXPTIME (upper bound)

In the previous section, we made crucial use of the property that the number of K rules on any branch of any cut-free proof of s in Lec+K is bounded by the modal depth $\text{md}(s)$ of s . It turns out that the same property holds for Lec+4. Let π be any cutfree proof in Lec+4.

First, suppose that s is some sequent in π that contains $\Box\alpha$ in its antecedent. By inspection, every rule instance in Lec+4 that concludes as s must have a premise whose antecedent contains $\Box\alpha$. It follows that there is a branch passing through s that leads to $\Box\alpha \Rightarrow \Box\alpha$ since this is the only way the branch can terminate. Since this initial sequent at the leaf of the proof contains a succedent copy of $\Box\alpha$ —we say that this occurrence has positive polarity with respect to the sequent—it follows from cut-freeness that every sequent on the branch from this leaf to the root contains an occurrence of $\Box\alpha$ with positive polarity (possibly occurring as a subformula under the usual reading e.g., in an implication $\alpha \rightarrow \beta$, the β preserves the polarity of $\alpha \rightarrow \beta$, and the α has the opposing polarity).

Now, suppose that s is the premise of a 4 rule. Then, it must have the following form: $\alpha_1, \dots, \alpha_n, \Box\alpha_1, \dots, \Box\alpha_n \Rightarrow \beta$. Call any $\Box\alpha_i$ that is not a subformula of $\Box\alpha_j$ for any $j \neq i$ as maximal. For any maximal $\Box\alpha_i$, the previous paragraph shows that s contains a subformula $\Box\alpha_i$ with positive polarity. Obviously, this subformula cannot occur within the negative polarity $\Box\alpha_i$, nor within any $\Box\alpha_j$ ($i \neq j$) as that would violate maximality of $\Box\alpha_i$. Thus, every maximal $\Box\alpha_i$ is a subformula of β (there is no where else that the positive $\Box\alpha_i$ can occur). Moreover, observe that any $\Box\gamma$ occurring as a subformula in the antecedent of s must be a subformula of some maximal $\Box\alpha_i$. We conclude that every boxed subformula in s is a subformula of β . Hence $\text{md}(s) = \text{md}(\beta) < \text{md}(\Box\beta) = \text{md}(s')$, where s' is the conclusion of the 4 rule.

Suppose that a branch in π rooted at s contains n occurrences of the 4 rule. Along this branch, let s_i be the premise of the i -th 4 rule from the root, and let s'_i be its conclusion. As the non-modal rule instances cannot increase the modal depth from conclusion to a premise:

$$\text{md}(s) \geq \text{md}(s'_1) > \text{md}(s_1) \geq \text{md}(s'_2) > \text{md}(s_2) \geq \dots \geq \text{md}(s'_n) > \text{md}(s_n) \geq 0$$

The strict inequalities in the above chain imply $n \leq \text{md}(s)$. We have established the following.

Lemma 8. *The number of 4 rules in any branch of a proof of s in Lec+4 is bounded by $\text{md}(s)$.*

The reduction from Lec+4 provability to EBVASS acceptance is similar to the reduction we used for Lec+K provability (Section 4), the difference lies in the simulation of the 4 rule in the former case. We add the unary rule $(q_{\Box\beta}^i, 0, q_{4,\beta}^{i+1})$; this is similar to what we did for Lec+K. We then add an auxiliary unary rule $(q_{4,\beta}^{i+1}, -e_{\Box\alpha}^i + e_{\Box\alpha}^{i+1} + e_{\alpha}^{i+1}, q_{4,\beta}^{i+1})$ which stays in this state and converts $e_{\Box\alpha}^i$ to $e_{\Box\alpha}^{i+1} + e_{\alpha}^{i+1}$ (see this as the $\Box\alpha$ in the conclusion becoming $\Box\alpha, \alpha$ in the premise). We add one further auxiliary unary rule $(q_{4,\beta}^{i+1}, 0, q_{\beta}^{i+1})$ that takes us out from the auxiliary state to q_{β}^{i+1} , so that the system is ready to simulate the next proof rule. We cannot explicitly check that every $e_{\Box\alpha}^i$ was converted to $e_{\Box\alpha}^{i+1} + e_{\alpha}^{i+1}$; as with the K rule, we add a superscript i on the counters to implicitly effect this check. In particular, if any $e_{\Box\alpha}^i$ was not decremented before moving out of the auxiliary state, it will have no further opportunity to decrement. This means that it will persist for the rest of the computation, making it impossible to reach an accepting configuration.

Formal definition. Let $s = (\Gamma \Rightarrow \beta)$ such that $\text{md}(s) = d$. We define the EBVASS C_s the same as B_s (see section 4), except we replace the rules that implement the K with the ones that implement the 4:

- The 4 rule is implemented via the unary rules

$$\begin{aligned} & \{(q_{\Box\beta}^i, 0, q_{4,\beta}^{i+1}) \mid \Box\beta \in \text{sub}(s), i \leq d\} \\ & \{(q_{4,\beta}^{i+1}, -e_{\Box\alpha}^i + e_{\Box\alpha}^{i+1} + e_{\alpha}^{i+1}, q_{4,\beta}^{i+1}) \mid \Box\alpha, \Box\beta \in \text{sub}(s), i \leq d\} \\ & \{(q_{4,\beta}^{i+1}, 0, q_{\beta}^{i+1}) \mid \Box\beta \in \text{sub}(s), i \leq d\} \end{aligned}$$

Let $\Gamma = \alpha_1, \dots, \alpha_k$ be a multiset of formulas. That there is an obvious bijection that maps Γ to the vector $\mathbf{c} = e_{\alpha_1} + \dots + e_{\alpha_k}$. In this case, we say that Γ is represented by $\mathbf{c}$. We define $e_{\Box\alpha}^i$ to be $e_{\Box\alpha}^i + e_{\alpha}^i$. Also define

$$\begin{aligned} \mathbf{c}^i &:= e_{\alpha_1}^i + \dots + e_{\alpha_k}^i \\ \Box\mathbf{c}^i &:= e_{\Box\alpha_1}^i + \dots + e_{\Box\alpha_k}^i \\ \Box\mathbf{c}^i &:= e_{\Box\alpha_1}^i + \dots + e_{\Box\alpha_k}^i \end{aligned}$$

Finally, define the function $\Theta^i(\alpha_1, \dots, \alpha_k \Rightarrow \beta) := (q_{\beta}^i, e_{\alpha_1}^i + \dots + e_{\alpha_k}^i)$.

5.1 From Lec+4 provability to EBVASS acceptance

Lemma 9. *Let π be a cut-free proof of $t = (\Gamma \Rightarrow \beta)$. Suppose that $s \in \pi$ and let i be the number of 4 applications from the root of π to s . Then, C_t accepts $\Theta^i(s)$.*

Proof. Induction on the height of the subproof rooted at s . The base case is when the height is 0, which can only happen when it is an axiom $\alpha \Rightarrow \alpha$. In this case, $\Theta^i(s) = (q_{\alpha}^i, e_{\Box\alpha}^i)$, which is an axiom of C_t , and so it is reachable.

For the inductive step, consider the last rule in this subproof. Most cases are the same as in Lemma 5, with the exception being the case for the 4 rule instead of the K rule.

The novel case is when the rule concluding s is the following 4 rule. Let s' be its premise.

$$\frac{\Gamma, \Box\Gamma \Rightarrow \beta}{\Box\Gamma \Rightarrow \Box\beta} 4$$

By IH, C_t accepts $\Theta^{i+1}(s') = (q_{\beta}^{i+1}, \Box\mathbf{c}^{i+1})$, where Γ is represented by the vector $\mathbf{c}$. Starting from $\Theta^i(s)$, our task is to reach $\Theta^{i+1}(s')$. First, apply the rule $r_1 = (q_{\Box\beta}^i, 0, q_{4,\beta}^{i+1})$:

$$\Theta^i(s) = (q_{\Box\beta}^i, \Box\mathbf{c}^i) \xrightarrow{r_1} (q_{4,\beta}^{i+1}, \Box\mathbf{c}^i)$$

Now, repeatedly apply $r_2 = (q_{4,\beta}^{i+1}, -e_{\Box\alpha}^i + e_{\Box\alpha}^{i+1} + e_{\alpha}^{i+1}, q_{4,\beta}^{i+1})$:

$$(q_{4,\beta}^{i+1}, \Box\mathbf{c}^i) \xrightarrow{r_2^*} (q_{4,\beta}^{i+1}, \Box\mathbf{c}^{i+1})$$

Applying $r_3 = (q_{4,\beta}^{i+1}, 0, q_{\beta}^{i+1})$, we have $(q_{4,\beta}^{i+1}, \Box\mathbf{c}^{i+1}) \xrightarrow{r_3} (q_{\beta}^{i+1}, \Box\mathbf{c}^{i+1}) = \Theta^{i+1}(s')$. $\square$

5.2 Reverse direction

Lemma 10. *Let s and t be sequents such that $\text{sub}(s) \subseteq \text{sub}(t)$. If C_t accepts $\Theta^i(s)$ for some $i \leq \text{md}(t)$, then s is provable.*

Proof. Induction on length of the computation. The base case is when the length is 0, it is an accepting configuration (q_α^i, e_α^i) . This configuration is the sequent $\alpha \Rightarrow \alpha$, which is an axiom and thus provable.

For the inductive step, we use case analysis on the first rule in the reachability witness. Most of the cases are the same as in **Lec+K** (see Lemma 6). The novel case is when the first rule is $r = (q_{\square\beta}^i, 0, q_{4,\beta}^{i+1})$, so

$$\Theta^i(s) = (q_{\square\beta}^i, \mathbf{c}^i) \xrightarrow{r} (q_{4,\beta}^{i+1}, \mathbf{c}^i)$$

The only rules that are applicable in state $q_{4,\beta}^{i+1}$ are $r_1 = (q_{4,\beta}^{i+1}, -e_{\square\alpha}^i + e_{\square\alpha}^{i+1} + e_\alpha^{i+1}, q_{4,\beta}^{i+1})$, $r_2 = (q_{4,\beta}^{i+1}, 0, q_\beta^{i+1})$, and the expansive rules. The only way to reach an axiom—as we must, since the hypothesis is acceptance—is to apply r_2 at some point, as that is the only way to get out of state $q_{4,\beta}^{i+1}$. Crucially, if r_2 were applied when $\mathbf{c}^i$ is not empty, then the non-empty coordinates would remain positive for the rest of the run; once again we could never reach an axiom, contradicting the acceptance hypothesis. It follows that, immediately prior to the r_2 application, we must have reached $(q_{4,\beta}^{i+1}, \boxtimes \mathbf{d}^{i+1})$ such that $\mathbf{c} \leq \mathbf{d}$ and $c_j = 0$ implies $d_j = 0$, where equality holds iff no expansion rule was applied. Now we have

$$(q_{4,\beta}^{i+1}, \boxtimes \mathbf{d}^{i+1}) \xrightarrow{r_2} (q_\beta^{i+1}, \boxtimes \mathbf{d}^{i+1}) = \Theta^{i+1}(s')$$

By IH, s' is provable. Now obtain the proof of s as follows.

$$\frac{\frac{s'}{\Gamma, \square\Gamma \Rightarrow \beta} c^*}{\square\Gamma \Rightarrow \square\beta} 4$$

□

Theorem 11. *Lec+4 provability is 2EXPTIME-complete.*

Proof. Lemmas 9 and 10 establish a polytime reduction $\langle t \rangle \mapsto \langle C_t, \Theta^i(t) \rangle$ from EBVASS acceptance to **Lec+4** acceptance, thus yielding the upper bound. The lower bound is via the trivial reduction from 2EXPTIME-complete **Lec** provability [16]. □

6 Lec+K deducibility is ACK-hard (lower bound)

We establish the **Lec+K** deducibility lower bound via a polynomial time reduction from expansive counter machine reachability to deducibility, extending a construction due to Urquhart [19].

Definition 12. *An expansive counter machine M is a tuple (k, Q, R, q_m) where $k \in \mathbb{N}$ is the number of counters, Q is a finite set of states, $R \subseteq Q \times (\{1, \dots, k\} \times \{++, --, =0?\}) \times Q$ is a finite set of rules that includes the expansive rule for each coordinate (defined below), and $q_m \in Q$ is the accepting state.*

A configuration of $M = (k, Q, R, q_m)$ is a tuple $(q, \mathbf{n})$, where $q \in Q$ is the current state and $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ records the counter values. The operational semantics is defined by transitions between configurations that are generated R and the expansive rules. Specifically,

- $(q_i, \mathbf{n}) \xrightarrow{n_j^{++}} (q_l, \mathbf{n}')$ if $(q_i, n_j^{++}, q_l) \in R$, $n'_j = n_j + 1$ and $n'_{j'} = n_{j'}$ for all $j \neq j'$.
- $(q_i, \mathbf{n}) \xrightarrow{n_j^{-}} (q_l, \mathbf{n}')$ if $(q_i, n_j^{-}, q_l) \in R$, $n'_j = n_j - 1 \geq 0$, and $n'_{j'} = n_{j'}$ for all $j \neq j'$.

- $(q_i, \mathbf{n}) \xrightarrow{n_j=0?} (q_l, \mathbf{n}')$ if $(q_i, n_j=0?, q_l) \in R$, $n'_j = 0 = n_j$, and $n'_{j'} = n_{j'}$ for all $j \neq j'$.
- $(q_i, \mathbf{n}) \xrightarrow{e} (q_l, \mathbf{n}')$ if $\mathbf{n} \leq \mathbf{n}'$ and $n_j = 0$ implies $n'_j = 0$ for every j (expansive rules).

The union of the first three cases is denoted $\xrightarrow{R}$. Let $\rightarrow^*$ be the reflexive transitive closure of $\xrightarrow{R}$, and $\xrightarrow{e}$. We say that an expansive CM M *accepts* a configuration $C = (q, \mathbf{n})$ if $(q, \mathbf{n}) \rightarrow^* (q_m, \mathbf{0})$. Without loss of generality, we may assume that there are no outgoing arrows from a configuration in the accepting state q_m . The following problem is ACK-hard [19, 17].

Problem 2. *Given an expansive counter machine M and configuration C , does M accept C ?*

Following Urquhart, we construct a finite set $theory(M)$ of sequents that code the rules of the expansive counter machine M , and a sequent $\Theta(C)$ whose antecedent codes the current configuration C , and whose succedent Q_m codes the accepting configuration. I.e.,

$$\Theta(q_i, n_1, \dots, n_k) := Q_i, p_1^{n_1}, \dots, p_k^{n_k} \Rightarrow Q_m \quad (1)$$

Here $Q_i = Q_j$ iff $i = j$, and $p_i = p_j$ iff $i = j$. The multiplicity of p_i in the antecedent corresponds to the number of tokens in the i -th counter in the current configuration. We will write the sequent simply as $Q_i, \mathbf{p}^{\mathbf{n}} \Rightarrow Q_m$ for brevity, where $\mathbf{p}^{\mathbf{n}}$ stands for $p_1^{n_1}, \dots, p_k^{n_k}$. We will also write $\mathbf{p}_{-j}^{\mathbf{n}}$ to mean that every p_j variable is omitted, i.e., $\mathbf{p}_{-j}^{\mathbf{n}}, p_j^{n_j} = \mathbf{p}^{\mathbf{n}}$.

Increments and decrement instructions are simulated by standard cuts exactly as done by Urquhart. The novelty here is simulating zero-test with standard cuts and the K rule (Urquhart was working in FLec, so he coded additive branching rules with the lattice disjunction instead). We can only apply the K rule if there are no non-boxed formulas present. When we want to simulate a zero-test on counter j , the idea is to use the cut-rule (read upwards) to prefix every formula other than p_j with a box. Then we can apply K iff there were no p_j 's present, i.e. this counter has value 0. Formally, define the set $theory(M)$ of sequents as follows:

- $(q_i \xrightarrow{n_j^{++}} q_l) \in R$ contributes $Q_i \Rightarrow Q_l \cdot p_j$
- $(q_i \xrightarrow{n_j^{--}} q_l) \in R$ contributes $Q_i, p_j \Rightarrow Q_l$
- $(q_i \xrightarrow{n_j=0?} q_l) \in R$ contributes the following sequents:
 - $Q_i \Rightarrow Q_{i,j} \cdot (\Box Q_m \rightarrow Q_m)$
 - $Q_{i,j}, p_{j'} \Rightarrow Q_{i,j} \cdot \Box p_{j'}$ for all $j' \neq j$
 - $Q_{i,j} \Rightarrow \Box Q_{i,j}^f$
 - $Q_{i,j}^f \Rightarrow Q_l$

We say that a sequent $\Gamma \Rightarrow \beta$ is a configuration if the sequent represents some configuration, i.e. there exists some q and $\mathbf{n}$ such that $\Theta(q, \mathbf{n})$ is $\Gamma \Rightarrow \beta$.

6.1 From ECM acceptance to Lec+K deducibility

The following lemma shows how to transform a reachability witness to a deduction. The novel case with respect to Urquhart is the simulation of zero-test using the K rule.

Lemma 13. *If an expansive CM M accepts $(q, \mathbf{n})$, then $\Theta(q, \mathbf{n})$ is deducible in $theory(M)$.*

Proof sketch. Induction on the length of the reachability witness.

Base case: The only possibility is that the starting configuration is itself the accepting configuration $(q_m, \mathbf{0})$. We have $\Theta(q_m, \mathbf{0}) = (Q_m \Rightarrow Q_m)$ which is obviously deducible.

Inductive case: Case analysis on the first instruction that could have been applied. Here and in Lemma 14, we show the novel case compared to Urquhart [19].

- The last instruction applied is a zero-test: $q_i \xrightarrow{n_j=0?} q_l$, in this case we get:

$$C = (q_i, n_1, \dots, n_j, \dots, n_k) \xrightarrow{n_j=0?} (q_l, n_1, \dots, n_j, \dots, n_k) = C'$$

where $n_j = 0$. We must show that $\Theta(C) = Q_i, \mathbf{p}^n \Rightarrow Q_m$ is deducible. First, proceed upwards from the latter sequent as follows (read the following diagram upwards from the root):

$$\frac{\frac{Q_i \Rightarrow Q_{i,j} \cdot (\Box Q_m \rightarrow Q_m)}{Q_i, \mathbf{p}^n \Rightarrow Q_m} \quad \frac{\frac{Q_{i,j}, \mathbf{p}^n \Rightarrow \Box Q_m \quad Q_m \Rightarrow Q_m}{Q_{i,j}, \Box Q_m \rightarrow Q_m, \mathbf{p}^n \Rightarrow Q_m} \rightarrow L \quad \frac{Q_{i,j}, \Box Q_m \rightarrow Q_m, \mathbf{p}^n \Rightarrow Q_m}{Q_{i,j} \cdot (\Box Q_m \rightarrow Q_m), \mathbf{p}^n \Rightarrow Q_m} \cdot L}{\frac{Q_{i,j}, \mathbf{p}^n \Rightarrow \Box Q_m \quad Q_{i,j} \cdot (\Box Q_m \rightarrow Q_m), \mathbf{p}^n \Rightarrow Q_m}{Q_i, \mathbf{p}^n \Rightarrow Q_m} cut} cut$$

Observe that the premise $Q_{i,j}, \mathbf{p}^n \Rightarrow \Box Q_m$ has the form $Q_{i,j}, \mathbf{p}^n, \Box \mathbf{p}^m \Rightarrow \Box Q_m$; at this first iteration, $\mathbf{m}$ is the zero vector, but this will change. Reading upwards from the root:

$$\frac{\frac{Q_{i,j}, \mathbf{p}^n, p_{j'}^{n_{j'}}, \Box \mathbf{p}^m, \Box p_{j'}^{m_{j'}+1} \Rightarrow \Box Q_m}{Q_{i,j}, \Box \mathbf{p}^n, p_{j'}^{n_{j'}}, \Box \mathbf{p}^m, \Box p_{j'}^{m_{j'}+1} \Rightarrow \Box Q_m} \cdot L \quad \frac{Q_{i,j}, p_{j'} \Rightarrow Q_{i,j} \cdot \Box p_{j'} \quad Q_{i,j}, \Box \mathbf{p}^n, p_{j'}^{n_{j'}}, \Box \mathbf{p}^m, \Box p_{j'}^{m_{j'}+1} \Rightarrow \Box Q_m}{Q_{i,j}, \Box \mathbf{p}^n, p_{j'}^{n_{j'}}, \Box \mathbf{p}^m, \Box p_{j'}^{m_{j'}+1} \Rightarrow \Box Q_m} cut}{Q_{i,j}, \Box \mathbf{p}^n, p_{j'}^{n_{j'}}, \Box \mathbf{p}^m, \Box p_{j'}^{m_{j'}+1} \Rightarrow \Box Q_m} cut$$

Compare the root to the right leaf. For some $j' \neq j$, we observe that $p_{j'}^{n_{j'}+1}$ has become $p_{j'}^{n_{j'}}$, and $p_{j'}^{m_{j'}+1}$ has become $p_{j'}^{m_{j'}}$. By repeating this action for every $j' \neq j$, we reach (upwards) the sequent $Q_{i,j}, \Box \mathbf{p}^n \Rightarrow \Box Q_m$. Notice that no p_j variables are present; this is because we are operating under the assumption $n_j = 0$. Proceed upwards from this sequent as follows.

$$\frac{\frac{Q_{i,j}^f \Rightarrow Q_l \quad Q_l, \mathbf{p}^n \Rightarrow Q_m}{Q_{i,j}^f, \mathbf{p}^n \Rightarrow Q_m} cut \quad \frac{Q_{i,j}^f, \mathbf{p}^n \Rightarrow Q_m}{\Box Q_{i,j}^f, \Box \mathbf{p}^n \Rightarrow \Box Q_m} K}{\frac{Q_{i,j} \Rightarrow \Box Q_{i,j}^f \quad \Box Q_{i,j}^f, \Box \mathbf{p}^n \Rightarrow \Box Q_m}{Q_{i,j}, \Box \mathbf{p}^n \Rightarrow \Box Q_m} cut} cut$$

Since $Q_l, \mathbf{p}^n \Rightarrow Q_m$ is deducible by IH, the claim is established. $\square$

6.2 Reverse direction

We now show how to extract a reachability witness from a suitable deduction. The former is more liberal than the latter but we will see that it is not significantly more liberal.

Lemma 14. *Let M be an expansive CM. If $\Theta(q, \mathbf{n})$ is deducible in $theory(M)$, then $(q, \mathbf{n})$ is accepted by M .*

Proof sketch. As $theory(M)$ is a regular theory, any deduction of $\Theta(q, \mathbf{n})$ from this theory implies the existence of a standard deduction of $\Theta(q, \mathbf{n})$ from $theory(M)$ (Lemma 3) i.e., every cut has as left premise a leaf from $theory(M)$. Induction on the height of this standard deduction.

Base case: By inspection, one of the following must hold.

- The sequent is a logical axiom. The only axiom in the image of Θ is $Q_m \Rightarrow Q_m$, representing the configuration $(q_m, \vec{0})$. The latter is obviously accepted by M .
- The sequent is in $theory(M)$. The only sequent that is a configuration is $Q_i, p_j \Rightarrow Q_m$, which is added when we have the instruction $q_i \xrightarrow{n_j--} q_m$. The corresponding configuration of $Q_i, p_j \Rightarrow Q_m$ is $(q_i, 0, \dots, 1, \dots, 0)$, where the 1 is at the j -th index. We can apply the instruction that we now know is present to get $(q_i, 0, \dots, 1, \dots, 0) \xrightarrow{n_j--} (q_m, \vec{0})$.

Inductive case: Case analysis on the last rule instance in the deduction. As the conclusion is in the image of Θ , it has the form $Q_i, \mathbf{p}^n \Rightarrow Q_m$. The interesting case is a cut-rule; in this case the left premise must be a leaf from $theory(M)$. The only possibilities are those with a Q_i in the antecedent. Increment and decrement sequents are quite straightforward so we consider the remaining possibility i.e., the left premise is the leaf $Q_i \Rightarrow Q_{i,j} \cdot (\Box Q_m \rightarrow Q_m)$:

$$\frac{\frac{Q_i \Rightarrow Q_{i,j} \cdot (\Box Q_m \rightarrow Q_m)}{Q_i, \mathbf{p}^n \Rightarrow Q_m} \quad \Box Q_{i,j} \cdot (\Box Q_m \rightarrow Q_m), \mathbf{p}^n \Rightarrow Q_m}{Q_i, \mathbf{p}^n \Rightarrow Q_m} cut$$

Observe that the right premise is not a logical axiom, nor does it belong to $theory(M)$. It cannot be the conclusion of a cut since the left premise of such a cut would have to be in $theory(M)$, and this is not possible by inspection. Therefore, the only rules that could have been applied above $\Box Q_{i,j} \cdot (\Box Q_m \rightarrow Q_m), \mathbf{p}^n \Rightarrow Q_m$ are contraction—this could only be on the p_i 's; otherwise there would be a branch upwards from here such every sequent on it has too many Q variables to reach a logical axiom or $theory(M)$ —and left fusion, and then left implication (once again, contraction on either $\Box Q_{i,j}$ or $(\Box Q_m \rightarrow Q_m)$ leads to an undeducible premise). This situation is presented below; note that in the $\rightarrow L$ rule application, the context cannot split in any other way. Indeed, if a p enters $Q_m \Rightarrow Q_m$, it cannot be removed since we assumed that q_m has no outgoing rules; if any of the other formulas enter $Q_m \Rightarrow Q_m$, then there would be too many Q variables.

$$\frac{\frac{\frac{Q_{i,j}, \mathbf{p}^{n''} \Rightarrow \Box Q_m \quad Q_m \Rightarrow Q_m}{Q_{i,j}, \Box Q_m \rightarrow Q_m, \mathbf{p}^{n''} \Rightarrow Q_m} \rightarrow L}{Q_{i,j}, \Box Q_m \rightarrow Q_m, \mathbf{p}^{n'} \Rightarrow Q_m} c^*}{\frac{Q_{i,j}, \Box Q_m \rightarrow Q_m, \mathbf{p}^{n'} \Rightarrow Q_m}{Q_{i,j} \cdot (\Box Q_m \rightarrow Q_m), \mathbf{p}^{n'} \Rightarrow Q_m} .L}{Q_{i,j} \cdot (\Box Q_m \rightarrow Q_m), \mathbf{p}^n \Rightarrow Q_m} c^*$$

Here $\mathbf{n}'' \geq \mathbf{n}' \geq \mathbf{n}$, and $n_i = 0 \rightarrow (n'_i = 0 \text{ and } n''_i = 0)$ for all i . The upper left sequent $Q_{i,j}, \mathbf{p}^{n''} \Rightarrow \Box Q_m$ has a box on the right side. The only way to have a box present in the sequent is if a K rule was applied above this at some point, so the only way for this formula to be deducible is for the K rule ultimately to be applied above it. First, let us write $Q_{i,j}, \mathbf{p}^{n''} \Rightarrow \Box Q_m$ as $Q_{i,j}, \mathbf{p}^s, \Box \mathbf{p}^t \Rightarrow \Box Q_m$. Although, $\mathbf{p}^s = \mathbf{p}^{n''}$ and $\mathbf{p}^t$ is empty at this moment, this general form because will be instantiated in the subsequent iterations.

The only rules that could conclude this sequent are contraction, cut with left premise leaf $Q_{i,j}, p_{j'} \Rightarrow Q_{i,j} \cdot \Box p_{j'}$ and left fusions after, or a cut with $Q_{i,j} \Rightarrow \Box Q_{i,j}^f$. The general setup for cuts with $Q_{i,j}, p_{j'} \Rightarrow Q_{i,j} \cdot \Box p_{j'}$ is the following ($\mathbf{s} \leq \mathbf{s}' \leq \mathbf{s}''$ and $\mathbf{t} \leq \mathbf{t}' \leq \mathbf{t}''$):

$$\begin{array}{c}
\frac{Q_{i,j}, \mathbf{p}_{-j'}^{s''}, p_{j'}^{s''}, \square \mathbf{p}_{-j'}^{t''}, \square p_{j'}^{t''+1} \Rightarrow \square Q_m}{Q_{i,j} \cdot \square p_{j'}, \mathbf{p}_{-j'}^{s''}, p_{j'}^{s''}, \square \mathbf{p}_{-j'}^{t''}, \square p_{j'}^{t''} \Rightarrow \square Q_m} \cdot L \\
\frac{Q_{i,j}, p_{j'} \Rightarrow Q_{i,j} \cdot \square p_{j'}}{Q_{i,j}, \mathbf{p}_{-j'}^{s'}, p_{j'}^{s'+1}, \square \mathbf{p}_{-j'}^{t'}, \square p_{j'}^{t'} \Rightarrow \square Q_m} c^* \\
\frac{Q_{i,j}, \mathbf{p}_{-j'}^{s'}, p_{j'}^{s'+1}, \square \mathbf{p}_{-j'}^{t'}, \square p_{j'}^{t'} \Rightarrow \square Q_m}{Q_{i,j}, \mathbf{p}_{-j'}^{s'}, p_{j'}^{s'+1}, \square \mathbf{p}_{-j'}^{t'}, \square p_{j'}^{t'} \Rightarrow \square Q_m} cut
\end{array}$$

Comparing the root and the premise of $\cdot L$, aside from the effects of contraction, observe that one copy of $p_{j'}$ ($j \neq j'$) has been converted to a copy of $\square p_{j'}$. This does not happen with p_j by design, since we do not have the sequent $Q_{i,j}, p_{j'} \Rightarrow Q_{i,j} \cdot \square p_{j'}$ with $j = j'$. Ultimately, we reach a sequent of the form: $Q_{i,j}, \square \mathbf{p}_{-j}^{\mathbf{l}}, p_j^{\mathbf{l}_j} \Rightarrow \square Q_m$ for some $\mathbf{l} \geq \mathbf{s} \geq \mathbf{n}$ and $n_i=0 \rightarrow l_i=0$ (if not every $p_{j'}$ with $j \neq j'$ was removed, this would block the K rule that we argue below must follow). At this stage, we may encounter a cut with $Q_{i,j} \Rightarrow \square Q_{i,j}^f$, as shown below. We observe that a cut with the latter could not have happened earlier, since it replaces $Q_{i,j}$ with $\square Q_{i,j}^f$ and then the removal of p_j in the above diagram would have been blocked.

$$\frac{Q_{i,j} \Rightarrow \square Q_{i,j}^f \quad \square Q_{i,j}^f, \square \mathbf{p}_{-j}^{\mathbf{l}}, p_j^{\mathbf{l}_j} \Rightarrow \square Q_m}{Q_{i,j}, \square \mathbf{p}_{-j}^{\mathbf{l}}, p_j^{\mathbf{l}_j} \Rightarrow \square Q_m} cut$$

No other split of the context yields deducible premises for the same reasons that we have already mentioned. Now, no non-modal logical rule could have been applied above $\square Q_{i,j}^f, \square \mathbf{p}_{-j}^{\mathbf{l}}, p_j^{m_j} \Rightarrow \square Q_m$ because no non-modal logical connective is visible. Moreover, no standard cut—and these are the only cuts that we can encounter in this standard deduction—could have been applied above it because there is no sequent in $theory(M)$ whose antecedent contains the formulas it would receive. Contraction may lead to $\mathbf{l}' \geq \mathbf{l}$ where $l_i=0 \rightarrow l'_i$ for all i . Since we cannot remain perpetually with a sequent of this form, some other rule must be applied above it. The only remaining possibility is the K; since its conclusion cannot contain any propositional variable, we conclude that $l'_j = 0$. Hence we have $\square \mathbf{p}_{-j}^{\mathbf{l}'}, p_j^{l'_j} = \square \mathbf{p}^{\mathbf{l}'}$, and the deduction appears as follows.

$$\frac{Q_{i,j}^f, \mathbf{p}^{\mathbf{l}'} \Rightarrow Q_m}{\square Q_{i,j}^f, \square \mathbf{p}^{\mathbf{l}'} \Rightarrow \square Q_m} K \\
\frac{\square Q_{i,j}^f, \square \mathbf{p}^{\mathbf{l}'} \Rightarrow \square Q_m}{\square Q_{i,j}^f, \square \mathbf{p}^{\mathbf{l}} \Rightarrow \square Q_m} c^*$$

By inspection, after some number of contractions above $Q_{i,j}^f, \mathbf{p}^{\mathbf{l}'} \Rightarrow Q_m$, we must encounter a cut with $Q_{i,j} \Rightarrow Q_l$ as follows, where $\mathbf{l}'' \geq \mathbf{l}'$ and $l'_i = 0 \rightarrow l''_i$ for all i . (once again, if any p -variables enter the left premise, the sequent will not be in $theory(M)$).

$$\frac{Q_{i,j}^f \Rightarrow Q_l \quad Q_l, \mathbf{p}^{\mathbf{l}''} \Rightarrow Q_m}{Q_{i,j}^f, \mathbf{p}^{\mathbf{l}''} \Rightarrow Q_m} cut \\
\frac{Q_{i,j}^f, \mathbf{p}^{\mathbf{l}''} \Rightarrow Q_m}{Q_{i,j}^f, \mathbf{p}^{\mathbf{l}'} \Rightarrow Q_m} c^*$$

The right premise of the above cut is $\Theta(C')$. From C' , use the corresponding zero test rule $q_i \xrightarrow{n_j=0?} q_l$ (the point of the argument above was to obtain $n_j = 0$) and contraction to get:

$$C = (q_i, n_1, \dots, n_k) \xrightarrow{n_j=0?} (q_l, n_1, \dots, n_k) \xrightarrow{c} (q_l, l''_1, \dots, l''_k) = C'$$

By induction hypothesis, C' is accepting, so C is also an accepting configuration. $\square$

Theorem 15. *Lec+K deducibility is ACK-complete*

Proof sketch. Lemmas 13 and 14 establish a polytime reduction from acceptance in expansive counter machines to Lec+K deducibility. As noted, the former is ACK-hard. The upper bound is due to Theorem 1. $\square$

7 Conclusion

Many of the results hold in the presence of T. Specifically, Lec+T provability and Lec+K+T provability are in 2EXPTIME. To see this, extend the reduction from provability to EBVASS acceptance, simulating the T as follows:

$$\{(q_{\beta}^i, -e_{\Box\alpha}^i + e_{\alpha}^i, q_{\beta}^i) \mid \alpha, \beta \in \text{sub}(s), i \leq j < d\}$$

Furthermore, since the weak form of the deduction theorem for Lec that is described in the Introduction applies also to Lec+T, we conclude that Lec+T deducibility is in 2EXPTIME.

In contrast to the above, the 2EXPTIME argument for Lec+4 provability does not extend to Lec+4+T provability: the Lec+4 case relied on Lemma 8 yet the analogous lemma for Lec+4+T does not hold i.e., we can no longer bound the number of 4 rules on a branch by the modal depth. Indeed, the first step in the proof of Lemma 8 traced every boxed formula in the antecedent to an axiom of the form $\Box\alpha \Rightarrow \Box\alpha$; this does not hold in the presence of the T rule since a boxed formula in the conclusion of this rule may lose its leading box in the premise.

The Lec+K deducibility lower bounding argument holds for Lec+K+T deducibility because the T rule necessarily makes a boxed formula principal, and we know exactly when such boxed formulas appear in the reduction: boxed formulas are present right before we apply the K rule. The T rule could not be applied on a boxed Q variable as that would make the premise undeducible. On the other hand, if the T rule unboxes a boxed p variable, that variable must later become boxed again since otherwise the K rule could not have been applied upwards.

We conjecture that Lec+4 deducibility, and Lec+4+T provability are ACK-hard (this would yield tight complexity since these problems are in ACK by Theorem 1). The idea could be to adapt the reduction from Lec+K to expansive CM acceptance, although there are several issues that would need to be resolved first: it seems that a gainy counter machine should be used instead of an expansive one—the premise of the 4 rule retains boxed copies of the counters, so the machine might have to increase a counter from 0 to 1. The candidate for $\text{theory}(M)$ that we have in mind for Lec+4 deducibility is also non-regular—it has some boxed formulas in the antecedent—so an analogue of Lemma 3 would need to be proved first.

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