Scheme representation for first-order logic

Steve Awodey and Spencer Breiner

Carnegie Mellon University, Pittsburgh, Pennsylvania, USA
awodey@cmu.edu
sbreiner@andrew.cmu.edu

Logical Schemes

Recall that every commutative ring R determines an affine scheme in algebraic geometry. This consists of two components: a topological space $\mathrm{Spec}(R)$ (the spectrum) and a sheaf of local rings $\mathcal{O}_R$ (the structure sheaf). In this way, a scheme encodes both geometric and algebraic data.

In this work, we present a construction of "logical schemes," geometric entities which represent logical theories in much the same way that algebraic schemes represent rings. These also involve two components: a semantic spectral space and a syntactic structure sheaf. As in the algebraic case, we can recover a theory from its scheme representation (up to a conservative completion) and the structure sheaf is local in a certain logical sense. From these affine pieces we can build up a 2-category of logical schemes which share some of the nice properties of algebraic schemes.

The "logical spectrum" of a first-order theory $\mathbb{T}$ is constructed from the semantics of $\mathbb{T}$. Points in $\mathrm{Spec}(\mathbb{T})$ are models supplemented by certain variable assignments or labellings. Satisfaction induces a topology on these points, much as in the Stone space construction for propositional logic. As in the algebraic case, the spectrum is *not* a Hausdorff space. Instead the topology incorporates model theoretic information; notably, the closure of a point (model) $M \in \mathrm{Spec}(\mathbb{T})$ can be interpreted as the set of model homomorphisms into M.

Every formula $\varphi(x_1, \dots, x_n)$ determines a "definable sheaf" $[\![\varphi]\!]$ over the spectrum. Over each model M, the fiber of $[\![\varphi]\!]$ is the definable set

$$\mathrm{stalk}_M([\![\varphi]\!]) = \varphi^M = \{\overline{a} \in |M|^n \mid M \models \varphi(\overline{a})\}.$$

The topology is defined by those terms t which satisfy $\varphi(t)$; each of these defines a section of the sheaf, sending $M \mapsto t^M \in \varphi^M$.

Although $[\![\varphi]\!]$ is nicely behaved, its subsheaves are not. The problem is that these may depend on details of the labellings which have no syntactic relevance. To "cancel out" this effect we appeal to $\mathbb{T}$-model isomorphisms. Specifically, we can topologize the isomorphisms between models, turning $\mathrm{Spec}(\mathbb{T})$ into a topological groupoid. Each definable sheaf $[\![\varphi]\!]$ is equivariant over this groupoid, which is just a fancy way of saying that an isomorphism $M \cong M'$ induces an isomorphism $\varphi^M \cong \varphi^{M'}$ for each definable set.

The pathological subsheaves, however, are not equivariant; any subsheaf $S \leq [\![\varphi]\!]$ which *is* equivariant must be a union of definable pieces $[\![\psi_i]\!]$, where $\psi_i(\overline{x}) \vdash \varphi(\overline{x})$. Moreover, S itself is definable just in case it is compact with respect to such covers. This reflects a deeper fact: $\mathrm{Spec}(\mathbb{T})$ gives a presentation of the classifying topos for $\mathbb{T}$. That is, a $\mathbb{T}$-model inside a topos $\mathcal{S}$ (e.g., **Sets**) is essentially the same as a geometric morphism from **Sets** into equivariant sheaves over $\mathrm{Spec}(\mathbb{T})$. The spectrum is based on a construction of Joyal & Tierney [10] which was later improved by Joyal & Moerdijk [9, 8], Butz & Moerdijk [6, 5] and Awodey & Forssell [7, 4].

The structure sheaf of a logical scheme depends on a sheaf representation for logical theories (construed as structured categories). The most familiar example of such a representation is

Grothendieck's observation that every commutative ring R is isomorphic to the ring of global sections of a certain sheaf over the Zariski topology on R. Together with a locality condition, this is essentially the construction of an affine algebraic scheme. Later it was shown by Lambek & Moerdijk [12], Lambek [13] and Awodey [2, 3] that toposes could also be represented as global sections of sheaves on certain (generalized) spaces. The structure sheaf $\mathcal{O}_{\mathbb{T}}$ is especially close in spirit to the last example.

Given a model $M \in \mathrm{Spec}(\mathbb{T})$, the stalk of $\mathcal{O}_{\mathbb{T}}$ over M consists of definable sets over M. That is, each point of $\mathcal{O}_{\mathbb{T}}$ is a defined by a triple $\langle M, \varphi(\overline{x}, \overline{y}), \overline{b}\rangle$ where M is a model, φ is a formula and $\overline{b} \in |M|^{\overline{y}}$. Two triples $\langle M, \varphi(\overline{x}, \overline{y}), \overline{b}\rangle$ and $\langle M', \psi(\overline{x'}, \overline{y'}), \overline{c}\rangle$ are equal just in case $M = M'$, $\overline{x} = \overline{x'}$ and for any model homomorphism $f : M \to N$ (or, in the classical case, elementary substructures $M \subseteq N$):

$$\left\{\overline{a} \in |N|^{\overline{x}} \mid N \models \varphi(\overline{a}, f(\overline{b}))\right\} = \left\{\overline{a} \in |N|^{\overline{x}} \mid N \models \psi(\overline{a}, f(\overline{c}))\right\}.$$

In particular, every formula $\varphi(\overline{x})$ determines a global section $\ulcorner\varphi\urcorner : M \mapsto \varphi^M$, and together with formal sums and quotients these are *all* of the equivariant sections. These additional sums and quotients corresponds to the (conservative) model-theoretic extension $\mathbb{T} \subseteq \mathbb{T}^{\mathrm{eq}}$. This gives the representation theorem alluded to above:

$$\mathrm{Eq}\Gamma(\mathcal{O}_{\mathbb{T}}) \simeq \mathbb{T}^{\mathrm{eq}}.$$

At this point it is worth noting a type-theoretic connection which allows us to recover the definable sheaves $[\![\varphi]\!]$ directly from the structure sheaf $\mathcal{O}_{\mathbb{T}}$. There is an auxilliary sheaf $\mathcal{U} = \mathcal{U}_{\mathbb{T}}$ whose points are triples $\langle M, \varphi^M, \overline{a} \in \varphi^M\rangle$. Here $M \in \mathrm{Spec}(\mathbb{T})$, φ^M is one of its definable subsets and $\overline{a}$ is an element of that subset.

This has an obvious projection $\mathcal{U} \to \mathcal{O}$, and one can show that $[\![\varphi]\!]$ is the pullback of $\mathcal{U}$ along $\ulcorner\varphi\urcorner$. This makes $\mathcal{U} \to \mathcal{O}$ into a sheaf of type-theoretic universes à la Streicher [14]. Such universes play a significant role in recent advances connecting homotopy and type theory [1], in particular playing a role in Voevodsky's univalence axiom [15, 11].

With the affine components defined, algebraic geometry provides a framework to study this type of object; a typical example is the handling of scheme morphisms. A map of theories is an interpretation $I : \mathbb{T} \to \mathbb{T}'$ (e.g., adding an axiom or extending the language). This induces a forgetful functor $I_\flat : \mathrm{Spec}(\mathbb{T}') \to \mathrm{Spec}(\mathbb{T})$, sending each $\mathbb{T}'$-model N to the $\mathbb{T}$-model which is the interpretation of I in N. If I is a linguistic extension then $I_\flat N$ is the usual reduct of N to $\mathcal{L}(\mathbb{T})$.

Moreover, I induces another morphism at the level of structure sheaves: $I^\sharp : I_\flat^* \mathcal{O}_{\mathbb{T}} \to \mathcal{O}_{\mathbb{T}'}$. On fibers, this sends each I^*N-definable set φ^{I^*N} to the N-definable set $(I\varphi)^N$ (the same set!). Equivalently, we can represent $I^\sharp$ as a map $\mathcal{O}_{\mathbb{T}} \to I_{\flat *}\mathcal{O}_{\mathbb{T}'}$ and, since $\Gamma_{\mathbb{T}'} \circ I_{\flat *} \cong \Gamma_{\mathbb{T}}$, the global sections of suffice to recover I:

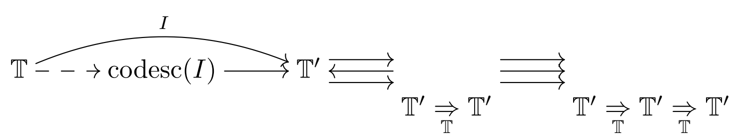
$$\Gamma_{\mathbb{T}}(I^\sharp) \cong I \;:\; \mathbb{T} = \Gamma_{\mathbb{T}}(\mathcal{O}_{\mathbb{T}}) \longrightarrow \Gamma_{\mathbb{T}}(I_{\flat *}\mathcal{O}_{\mathbb{T}'}) \cong \Gamma_{\mathbb{T}'}(\mathcal{O}_{\mathbb{T}'}) \cong \mathbb{T}'.$$

This addresses a significant difficulty in Forssell & Awodey's first-order logical duality [4]: identifying which homomorphisms between spectra originate syntactically. This problem is non-existent for schemes: without a syntactic map at the level of structure sheaves, there is no scheme morphism.

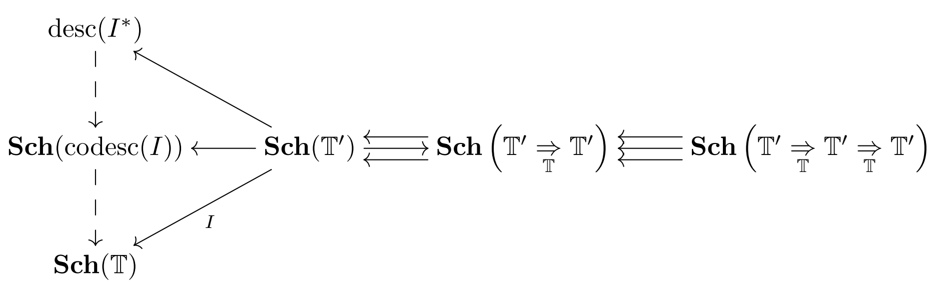
As with the morphisms, the necessary definitions to proceed from affine schemes to the general case follow the same rubric as algebraic geometry. There are analogs of locally ringed spaces, gluings (properly generalized to groupoidal spectra) and coverings by affine pieces. Importantly, the equivariant global sections functor presents (the opposite of) the 2-category of theories as a reflective subcategory of schemes. This allows us to construct limits of affine schemes using colimits of theories. This mirrors the algebraic situation, where the polynomial ring $\mathbb{Z}[x]$ represents the affine line and its coproduct $\mathbb{Z}[x,y] \cong \mathbb{Z}[x] + \mathbb{Z}[y]$ represents the plane. Via affine covers, one can use this to compute any finite 2-limits for any logical schemes.

Current research aims to connect the algebraic formalism back to logic, particularly model theory. Given the affine scheme construction, one natural question to consider is the existence and utility of *non*affine schemes. Model theory provides a good class of potential candidates in the form of pseudoelementary classes (PECs); such a class consists of those $\mathbb{T}$-models which are reducts of some extension $I : \mathbb{T} \to \mathbb{T}'$. On one hand it is well-known that these classes may fail to be axiomatizable; on the other, we know that the $I^* : \mathrm{Spec}(\mathbb{T}') \to \mathrm{Spec}(\mathbb{T})$ acts by sending a $\mathbb{T}'$-model to its reduct. This suggest that PECs may arise as a sort of image factorization among schemes.

In the context of schemes, descent is the appropriate generalization for image factorization. Suppose that we have an interpretation of theories $I : \mathbb{T} \to \mathbb{T}'$. Categorically construed, theories are closed under 2-limits and colimits, allowing us to build a coresolution of I (lax pushouts) and a codescent category (a 2-categorical "equalizer"):



Limits of schemes are computed using colimits of theories, but colimits of schemes are typically different from the associated limits in theories. This means that descent objects in schemes (when they exist) induce a further factorization of I^*:



Moreover, these factorizations seem related to existing topos-theoretic constructions, specifically the quotient/conservative and hyperconnected/localic factorizations systems.

References

[1] S. Awodey. Type theory and homotopy. In *Epistemology versus Ontology*, pages 183–201. Springer, 2012.

[2] Steve Awodey. *Logic in Topoi: Functorial Semantics for Higher-Order Logic*. PhD thesis, University of Chicago, 1997. `http://www.andrew.cmu.edu/user/awodey/thesis/thesis.ps.gz`.

[3] Steve Awodey. Sheaf representation for topoi. *Journal of Pure and Applied Algebra*, 145:107–121, 2000.

[4] Steve Awodey and Henrik Forssell. First-order logical duality. *Annals of Pure and Applied Logic*, 164:319–348, March 2012.

[5] C. Butz and I. Moerdijk. Representing topoi by topological groupoids. *Journal of Pure and Applied Algebra*, 130(3):223–235, 1998.

[6] Carsten Butz. *Logical and Cohomological Aspects of the Space of Points of a Topos*. PhD thesis, Universiteit Utrecht, 1991.

[7] Henrik Forssell. *First-Order Logical Duality*. PhD thesis, Carnegie Mellon University, 2008.

[8] A. Joyal and I. Moerdijk. Toposes are cohomologically equivalent to spaces. *American Journal of Mathematics*, pages 87–95, 1990.

[9] A. Joyal and I. Moerdijk. Toposes as homotopy groupoids. *Advances in Mathematics*, 80(1):22–38, 1990.

[10] A. Joyal and M. Tierney. *An extension of the Galois theory of Grothendieck*. American Mathematical Society, 1984.

[11] C. Kapulkin, P.L.F. Lumsdaine, and V. Voevodsky. Univalence in simplicial sets. 2012.

[12] J. Lambek and I. Moerdijk. Two sheaf representations of elementary toposes. *Studies in Logic and the Foundations of Mathematics*, 110:275–295, 1982.

[13] Joachim Lambek. On the sheaf of possible worlds. In Jiri Adamek and Saunders MacLane, editors, *Categorical Topology and its Relations to Analysis, Algebra and Combinatorics*, pages 36–53. World Science Publishing, 1989.

[14] Thomas Streicher. Universes in a topos. In Laura Crosilla and Peter Schuster, editors, *From Sets and Types to Topology and Analysis*, number 48 in Oxford Logic Guides, pages 78–90. Oxford University Press, 2005. `http://www.mathematik.tu-darmstadt.de/~streicher/NOTES/UniTop.ps.gz`.

[15] V. Voevodsky. Univalent foundations project. 2010.