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Common Knowledge in an Epistemic Logic with Hypotheses*

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Abstract

We extend epistemic logic $S5^r$ for reasoning about knowledge under hypotheses with distributive knowledge operator. This extension gives possibility to express distributive knowledge of agents with different background assumptions. The logic is important in computer science since it models agents behavior which already have some equipped knowledge. Extension with distributive knowledge shows to be extremely interesting since knowledge of an arbitrary agent whose epistemic capacity corresponds to any system between S4 and S5 under some restrictions can be modeled as distributive knowledge of agents with certain background knowledge. We present an axiomatization of the logic and prove Kripke completeness and decidability results.

1 Introduction

In this paper we take further study a of the logic of hypotheses $\mathsf{S5}^r$, enriched with epistemic operators. In recent work [9] we have studied extension of the logic $\mathsf{S5}^r$ by common knowledge operator. In the current work we study enrichment of the logic by distributive knowledge operator. The logic $\mathsf{S5}^r$ has been introduced in [7]. It is an extension of the epistemic logic S5 with a modal operator '$[\cdot]$' that can be parameterized with a hypothesis. The operator can be described as relative necessity, a notion already used by Chellas to describe conditionality [1].

The modal operator $[\varphi]$ stands to represent the knowledge state where the hypothesis φ is assumed. The formula $[\varphi]\psi$ can be read as 'under the hypothesis φ, the agent knows ψ'. If φ happens to be true at the current world and the agent knows that φ implies ψ, then the agent knows ψ; otherwise, i.e., if φ is false, the agent knows only what it would know anyway, i.e. without any assumptions. This way of reading of the modality $[\varphi]\psi$ is supported by the following reduction axiom $[\varphi]_K\,\psi \leftrightarrow [\top]_K\,\psi \vee (\varphi \wedge [\top]_K\,(\varphi \to \psi))$

For instance, consider a simple dice game, where the game is won if, and only if, a three or a six has been rolled. The formula '$[\,\top\,]\,(\mathsf{three} \vee \mathsf{six} \leftrightarrow \mathsf{win})$' states that the agent knows this rule. The parameter '$\top$' of the box-modality stands for the fact that no hypothesis is being adopted

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by the agent. Suppose that the dice is rolled under a cup, so that the rolled number of points is concealed from the agent. Consequently, as long as the dice remains concealed, the agent does not know whether or not the game is won. This can be described by the formula '$\neg [\top] \mathsf{win}$'. However, the agent knows that the game is won under the hypothesis that a six has been rolled: '$[\mathsf{six}] \mathsf{win}$'. We can distinguish two situations: one, where the hypothesis is correct, i.e., a six has been rolled; and another one, where it is false, i.e., the dice shows a number between one to five. In the former situation, the game is won and we have that the formula holds true. In case the hypothesis is in fact wrong, the formula is not necessarily true. Irrespective of the hypotheses held by the agent, the game may still be won provided that a three has been rolled. In all other cases, the game is lost. This is different to ordinary implications, which are true whenever the premise is false or the consequent is true.

In this paper, we continue our investigation of $\mathsf{S5}^r$. We consider the extension of $\mathsf{S5}^r$ with operators for distributed knowledge. Distributed knowledge is a standard notion in epistemic logic [2]. Distributed knowledge of a group of agents equals what a single agent knows who knows everything what each member of the group knows. For instance, if agent a knows p and agent b knows $p \to q$, then the distributed knowledge between a and b includes q, even though neither of them might know q individually. The notion of distributed knowledge is relevant for describing and reasoning about the combined knowledge of agents in a distributed system; see, e.g., [3]. Agents communicate with each other to combine their knowledge. Thus the notion of distributed knowledge is also central to communication protocols and relevant to reasoning about speech acts [4, 6]. We may think of distributed hypotheses as the result of combining incoming information from several sources. However, the truthfulness of the incoming information is not assumed. We demonstrate another way to think about distributed hypotheses by using $\mathsf{S5}^r$ to represent the knowledge of an agent whose knowledge capacity can be characterized by any modal system between S4 and S5. As a main result we show that $\mathsf{S5D}^r$ ($\mathsf{S5}^r$ extended with distributive knowledge operator) is Kripke complete and decidable. For proving Kripke completeness results we use technique developed [8]

The paper is organized as follows: In the next section, we briefly recall basic definitions on modal logic, Kriple semantics, the standard translation to first-order logic. Next we review a technique from [8] for obtaining Kripke completeness results for certain extensions of a modal logic. In Section 3, we show that the techniques are also applicable to $\mathsf{S5D}^r$. Additionally we show how knowledge of an agent can be represented as a knowledge under distributed hypotheses, where the agents' knowledge corresponds to any system between S4 and S5.

2 Preliminaries

In this section, we briefly review modal logic. Moreover, we introduce a simple technique for obtaining Kripke completeness.This technique in detail can be seen in [8].

Let $\langle \Pi, M \rangle$ be a *signature* consisting of countable sets Π and M of symbols for propositions and modalities, respectively. The *propositional modal language* $\mathcal{L}$ for this signature consists of formulas φ that are built up inductively according to the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_m \varphi,$$

where p ranges over proposition symbols in Π and m over modality symbols in M. The logical symbols '$\top$' and '$\bot$', and the additional connectives such as '$\vee$', '$\to$' and '$\leftrightarrow$' and the dual modalities '$\Diamond_m$' with $m \in M$ are defined as usual, i.e.: $\top := p \vee \neg p$ for some atomic proposition p; $\bot := \neg\top$; $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$; $\varphi \to \psi := \neg\varphi \vee \psi$; $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$; and $\Diamond_m \varphi := \neg\Box_m \neg\varphi$.

A subset L of the propositional modal language $\mathcal{L}$ is a *modal logic* iff it contains all propositional tautologies, is closed under substitution, modus ponens and modal replacement (MREP) $\frac{p\leftrightarrow q}{\Box_m p\leftrightarrow\Box_m q}$, for $m \in M$. The modal logic L is called *normal* if it contains the formulas (K) $\Box_m(p \to q) \to (\Box_m p \to \Box_m q)$ and is closed under (NEC) $\frac{p}{\Box_m p}$. The smallest normal modal logic is commonly denoted with K.

2.1 Kripke Semantics

The relational semantics for the propositional modal language $\mathcal{L}$ is based on Kripke structures for the signature $\langle\Pi, M\rangle$ of $\mathcal{L}$. Formally, an *M-frame* (or Kripke frame) is a tuple $\mathfrak{F} = (W, \{R_m\}_{m\in M})$, where W is a non-empty set of worlds and $R_m \subseteq W^2$ a binary relation over W, for every $m \in M$. A *Kripke model for* $\langle\Pi, M\rangle$ is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ consisting of a Kripke frame $\mathfrak{F} = (W, \{R_m\}_{m\in M})$ together with a valuation function $V : \Pi \to 2^W$ assigning to every proposition p in Π a set $V(p)$ of worlds. A Kripke model $\mathfrak{M} = (\mathfrak{F}, V)$ is said to be *based on the frame* $\mathfrak{F}$.

An interpretation of formulas from $\mathcal{L}$ is given by means of a *satisfaction relation* '$\models$', which is a binary relation between pointed models and formulas. A pointed model is a pair $\langle\mathfrak{M}, w\rangle$, where $\mathfrak{M} = (W, \{R_m\}_{m\in M}, V)$ is a Kripke model and w a world from W. The satisfaction relation is defined inductively on the structure of formulas φ as:

- $\langle\mathfrak{M}, w\rangle \models p$ iff $w \in V(p)$;
- $\langle\mathfrak{M}, w\rangle \models \neg\psi$ iff $\langle\mathfrak{M}, w\rangle \not\models \psi$;
- $\langle\mathfrak{M}, w\rangle \models \psi \wedge \chi$ iff $\langle\mathfrak{M}, w\rangle \models \psi$ and $\langle\mathfrak{M}, w\rangle \models \chi$;
- $\langle\mathfrak{M}, w\rangle \models \Box_m\psi$ iff for all $v \in W$ with $(w, v) \in R_m$, $\langle\mathfrak{M}, v\rangle \models \psi$.

A formula φ is said to be *true* at w in $\mathfrak{M}$ iff $\langle\mathfrak{M}, w\rangle \models \varphi$; φ is *satisfiable* iff there is a pointed model $\langle\mathfrak{M}, w\rangle$ at which it is true; φ is *valid in* $\mathfrak{M}$ (written '$\mathfrak{M} \models \varphi$') iff $\langle\mathfrak{M}, w\rangle \models \varphi$ for all w in $\mathfrak{M}$; φ is *valid* on $\mathfrak{F}$ (written '$\mathfrak{F} \models \varphi$') iff φ is valid in all models based on $\mathfrak{F}$; and φ is *valid* in the class $\mathcal{C}$ of Kripke frames (written '$\models_{\mathcal{C}} \varphi$') iff it is valid in every Kripke frame from $\mathcal{C}$.

The set of $\mathcal{L}$-formulas that are valid in a class $\mathcal{C}$ of Kripe frames is called the *$\mathcal{L}$-theory* $\mathsf{Th}_{\mathcal{L}}(\mathcal{C})$ of $\mathcal{C}$, i.e.:

$$\mathsf{Th}_{\mathcal{L}}(\mathcal{C}) := \{\, \varphi \in \mathcal{L} \mid \text{for every } \mathfrak{F} \text{ from } \mathcal{C},\ \varphi \text{ is valid in } \mathfrak{F} \,\}.$$

A modal logic L is said to be *Kripke complete w.r.t.* $\mathcal{C}$ iff $L \supseteq \mathsf{Th}_{\mathcal{L}}(\mathcal{C})$, and L is said to be *sound w.r.t.* $\mathcal{C}$ iff $L \subseteq \mathsf{Th}_{\mathcal{L}}(\mathcal{C})$.

2.2 Standard Translation

The relationship to first-order logic is made precise by the so-called *standard translation* $\mathrm{ST}(\cdot)$, which assigns to a modal formula φ a corresponding first-order formula $\mathrm{ST}_x(\varphi)$ with one free variable x. The signature of the first-order language contains unary predicate symbols P and binary predicate symbols R_m, one P for every $p \in \Pi$ and one R_m for every $m \in M$. The translation function $\mathrm{ST}(\cdot)$ is inductively defined as follows:

$$\begin{array}{rcl}
\mathrm{ST}_x(p) & := & P(x) \\
\mathrm{ST}_x(\neg\varphi) & := & \neg\mathrm{ST}_x(\varphi) \\
\mathrm{ST}_x(\varphi \wedge \psi) & := & \mathrm{ST}_x(\varphi) \wedge \mathrm{ST}_x(\psi) \\
\mathrm{ST}_x(\Box_m\varphi) & := & \forall y(R_m(x, y) \to \mathrm{ST}_y(\varphi))
\end{array}$$

where y is a fresh variable for every occurrence of a box-modality.

A Kripke structure $\mathfrak{M} = (W, \{R_m\}_{m \in M}, V)$ for $\langle \Pi, M \rangle$ can be seen as a first-order structure interpreting the formula $\mathrm{ST}_x(\varphi)$. While a predicate symbol R_m is interpreted using the binary relation R_m over W that is interpreting the modality m in M, a predicate symbol P is interpreted as the subset $V(p)$ of W, where p is the proposition symbol from Π that corresponds to P. Neither constants nor function symbols are introduced by the standard translation. In the first-order structure $\mathfrak{M}$, however, we introduce a dedicated constant c_w for every world $w \in W$ and we interpret c_w as w. At the level of pointed models $\langle \mathfrak{M}, w \rangle$, the relationship between φ and $\mathrm{ST}_x(\varphi)$ is such that:

$$\langle \mathfrak{M}, w \rangle \models \varphi \text{ iff } \mathfrak{M} \models \mathrm{ST}_x(\varphi)[x \mapsto c_w],$$

where $[x \mapsto c_w]$ substitutes every occurrence of the free variable x in $\mathrm{ST}_x(\varphi)$ with the constant c_w. Note that $\mathrm{ST}_x(\varphi)[x \mapsto c_w]$ is a sentence, i.e. a first-order formula without free variables.

When considering the notion of validity on frames $\mathfrak{F}$, we have that φ corresponds to the monadic second-order formula $\forall \vec{P} \forall x \, \mathrm{ST}_x(\varphi)$ as follows:

$$\mathfrak{F} \models \varphi(\vec{p}) \text{ iff } \mathfrak{F} \models \forall \vec{P} \forall x \, \mathrm{ST}_x(\varphi),$$

where $\vec{p}$ are the propositions from Π that occur in φ, and $\vec{P}$ are the corresponding unary predicates.

2.3 Completeness by Modal Definitions

In [8] we introduced a technique on how to obtain Kripke completeness w.r.t. a specific class of Kripke structures for certain extensions of complete modal logics. We apply this technique to extensions of the modal logic $\mathsf{S5}$.

By *extending a modal logic L with a formula φ* we mean obtaining a modal logic L' as a set of formulas that is minimal w.r.t. $\subseteq$, that contains all tautologies over the symbols for propositions occurring in $L \cup \{\varphi\}$, that contains all formulas from $L \cup \{\varphi\}$ and that is closed under substitution, modus-ponens and modal replacement. Clearly $L \cup \{\varphi\}$ is not necessarily a modal logic. Moreover, an extension of a modal logic that is Kripke complete w.r.t. a class $\mathcal{C}$ of models is not necessarily complete w.r.t. $\mathcal{C}$ itself nor any other class of models. We are interested in studying formulas of a specific form (modal definitions) that, when used to extend a modal logic, yield a modal logic that is complete w.r.t. a specific class of models. Let $\mathcal{L}$ be a propositional modal language over the signature $\langle \Pi, M \rangle$. Let $\varphi(\vec{p})$ be a formula in $\mathcal{L}$, where $\vec{p}$ are the propositions occurring in φ. Let '+' be a fresh symbol for a unary modality not in M, and $\boxplus$ the box-version of this modality. A *modal definition* in $\mathcal{L}$ is a formula of the form

$$\boxplus p \leftrightarrow \varphi(\vec{p}),$$

where $\vec{p}$ contains p. The box-modality $\boxplus$ is defined in terms of a modal formula in which $\boxplus$ does not occur. Notice that the modal definition $\boxplus p \leftrightarrow \varphi(\vec{p})$ itself is a formula in the propositional modal language over the extended signature $\langle \Pi, M \cup \{+\} \rangle$. We only consider $+$ to be a unary modality symbol. Moreover, we will only consider the modal definitions for the box-version of $+$. The results for the dual modality can be obtained in a similar way.

In this paper, we only consider modal definitions $\boxplus p \leftrightarrow \varphi(\vec{p})$, where the box-modality $\boxplus$ does not occur in $\varphi(\vec{p})$.

A modal definition is interpreted in models $\mathfrak{M} = (\mathfrak{F}, V)$ that are based on $M \cup \{+\}$-frames

$\mathfrak{F} = (W, \{R_m\}_{m \in M} \cup \{R_+\})$, i.e., frames that are extended with a binary relation R_+ to interpret the new box-modality $\boxplus$. The semantics of $\boxplus$ can be defined in the usual way as for any other box-modality:

- $\langle \mathfrak{M}, w \rangle \models \boxplus\psi$ iff for all $v \in W$ with $(w, v) \in R_+$, it holds that $\langle \mathfrak{M}, v \rangle \models \psi$.

We want to interpret $\boxplus$ as specified in the modal logic L' obtained from the modal logic L extended with a modal definition of $\boxplus$. To this end, we have to confine ourselves to the models from $\mathcal{C}(L')$, i.e., all models from $\mathcal{K}_{\langle \Pi, M \cup \{+\} \rangle}$ in which all formulas of L' are valid. It is now interesting to investigate the relationship between the modal definition of $\boxplus$ and the properties of the relation R_+ in the models from $\mathcal{C}(L')$.

Example 1. *Let $\mathcal{L}$ be a propositional modal language over a signature $\langle \Pi, M \rangle$. Additionally, let '$+$' be a fresh symbol for a modality not in M. Finally, let $L \subseteq \mathcal{L}$ be a modal logic.*

The modal definition $\alpha_1 = \boxplus p \leftrightarrow p$ yields that R_+ is the identity relation. This can be seen as follows. Obtain the modal logic L_1 by extending L with α_1. One can see that class of frames for L_1 is the class of frames for L extended with the relation R_+ being the identity relation.

Another simple example of a modal definition is $\boxplus p \leftrightarrow \Box_m p$, for some $m \in M$. Here we have that R_+ equals R_m in every model. Consider two more examples: $\boxplus p \leftrightarrow p \vee \neg p$ and $\boxplus p \leftrightarrow p \wedge \neg p$. In the former case, R_+ is the empty relation, whereas in the latter case the modal definition does not yield any relation.

As the examples show, not all modal definitions yield a relational semantics for the logic extended with the newly defined modality. Taking the standard translation of a formula φ that is used in a definition $\boxplus p \leftrightarrow \varphi(\vec{p})$ results in the second-order formula $\forall \vec{P} \forall x \, \mathrm{ST}_x(\varphi)$, where the predicates in $\vec{P}$ correspond to the propositional variables in $\vec{p}$. We are interested in elementary formulas, i.e., those formulas φ for which there exists a first-order formula that is equivalent to the second-order formula $\forall \vec{P} \forall x \, \mathrm{ST}_x(\varphi)$, that additionally yield a relational semantics for the new modality $+$. It is a non-trivial problem to give a syntactic characterization of such formulas φ that are suitable for defining fresh modalities.

To start tackling this problem, we introduce the notion of a 'relational modal definition'.

Definition 1. *Let $\mathcal{L}$ be a propositional modal language over the signature $\langle \Pi, M \rangle$. Let $\varphi(p, p_1, \ldots, p_n)$ with $n \geq 0$ be a formula in $\mathcal{L}$, where $p, p_1, \ldots, p_n$ are the propositions occurring in φ. Let '$+$' be a fresh symbol for a unary modality not in M, and $\boxplus$ the box-version of this modality.*

A modal definition $\boxplus p \leftrightarrow \varphi(p, p_1, \ldots, p_n)$ is called a relational modal definition *if there exists a first-order formula $\Psi_+(x, y)$ with two free variables x and y using only predicates that occur in $\mathrm{ST}_x(\varphi(p, p_1, \ldots, p_n))$ such that for every $\psi \in \mathcal{L}$, it holds that for all pointed models $\langle \mathfrak{M}, w \rangle$, $\mathfrak{M} \models (\forall y)(\Psi_+(x, y) \to \mathrm{ST}_y(\psi))[x \mapsto c_w]$ iff $\mathfrak{M} \models \mathrm{ST}_x(\varphi(\psi, p_1, \ldots, p_n))[x \mapsto c_w]$.*

Example 2. *Let us consider modal logic K extended with a new modality $\boxplus$. A formula $\boxplus p \leftrightarrow \Box p \wedge p$ is a relational modal definition. Indeed, for every pointed model $\langle \mathfrak{M}, w \rangle$, it holds that $\mathfrak{M} \models ((\forall y)(xRy \to P(y)) \wedge P(x))[x \mapsto c_w]$ iff $(\forall y)(\Psi(x, y) \to P(y))[x \mapsto c_w]$, where $\Psi(x, y)$ is the formula $(xRy) \vee (x = y)$.*

We note that elementarity is neither a sufficient nor a necessary condition for modal formulas being suitable for a relational modal definition; see, e.g., the reduction axiom for $\mathsf{S5}^r$ in the following section which yields a relational modal definition despite it being non-elementary.

Let $\Psi_+(x, y)$ be the first-order formula with two free variables x and y corresponding to a relational modal definition. Given a model $\mathfrak{M} = (\mathfrak{F}, V)$ with $\mathfrak{F} = (W, \{R_m\}_{m \in M})$, we uniquely construct the model $\mathfrak{M}_+ = (\mathfrak{F}_+, V)$, where the underlying frame $\mathfrak{F}_+$ is obtained from $\mathfrak{F}$ by adding the binary relation $R_+ \subseteq W \times W$ defined as:

$$(v, w) \in R_+ \text{ iff } \mathfrak{M} \models \Psi_+(x, y)[x \mapsto c_v, y \mapsto c_w].$$

For a class $\mathcal{C}$ of models, we denote with $\mathcal{C}_+$ the class consisting of the models $\mathfrak{M}_+$, where $\mathfrak{M}$ ranges over the models in $\mathcal{C}$.

Formulas from the extended language $\mathcal{L}_+$ can be translated to formulas in $\mathcal{L}$ in a straightforward way.

Definition 2. *Let $\mathcal{L}$ and $\mathcal{L}_+$ be propositional modal languages over the signatures $\langle \Pi, M\rangle$ and $\langle \Pi, M \cup \{+\}\rangle$, respectively, where $+$ is a fresh unary modality not in M. The translation function $^*\colon \mathcal{L}^+ \to \mathcal{L}$ for the modal definition $\boxplus p \leftrightarrow \varphi_+(p, p_1, \ldots, p_n)$ is inductively defined as follows, where m ranges over M:*

$$\begin{array}{rcl} p^* & := & p \\ (\varphi \vee \psi)^* & := & \varphi^* \vee \psi^* \\ (\neg\varphi)^* & := & \neg\varphi^* \\ (\Box_m \varphi)^* & := & \Box_m \varphi^* \\ (\boxplus\psi)^* & := & \varphi_+(\psi^*, p_1, \ldots, p_n) \end{array}$$

The following theorem shows the intended completeness technique.

Theorem 1 ([8])**.** *Let $\mathcal{L}$ and $\mathcal{L}_+$ be propositional modal languages over the signatures $\langle \Pi, M\rangle$ and $\langle \Pi, M \cup \{+\}\rangle$, respectively, where $+$ is a fresh unary modality not in M. Let $L \subseteq \mathcal{L}$ be a normal modal logic that is sound and complete w.r.t. a class $\mathcal{F}$ of Kripke frames. Obtain $L_+ \subseteq \mathcal{L}_+$ from L by adding a relational modal definition $\boxplus p \leftrightarrow \varphi(p_1, \ldots, p_n)$ as an only axiom schema for $\boxplus$. Then the logic L_+ is sound and complete w.r.t. the class $\mathcal{F}_+$.*

3 The Modal Logic $\mathsf{S5D}^r$

In this section, we introduce the modal logic $\mathsf{S5D}^r$ which is an extension of $\mathsf{S5}^r$ with modalities for distributed hypotheses that are analogous to modalities for distributed knowledge. Distributed knowledge in modal logic is a well-known notion; standard references include [2, 5] and for a more recent discussion, see [4, 6]. We prove the completeness and decidability results for this logic. We show how distributed hypotheses can be used to represent the knowledge of an agent whose epistemic capacity corresponds to any system containing S4.

Let Π be a countably infinite set of atomic propositions. Formulas φ of $\mathsf{S5D}^r$ are defined inductively over Π by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid [\varphi]_K\, \varphi \mid [\Phi]_D\, \varphi,$$

where p ranges over atomic propositions in Π, and Φ over finite sets of $\mathsf{S5D}^r$-formulas.

Formulas of $\mathsf{S5D}^r$ are evaluated in basic structures as well. The operators $[\Phi]_D$ are necessities depending on the formulas in Φ. The semantics of $[\Phi]_D$ is based on the relations R_φ, where $\varphi \in \Phi$, as follows. Let $\mathfrak{M} = (W, V)$ be a basic structure. The logical consequence relation '$\models$' and the relations R for formulas of $\mathsf{S5D}^r$ are defined as for $\mathsf{S5}^r$ but extended with the following clauses: For all $\mathsf{S5D}^r$-formulas ψ and all finite sets Φ of $\mathsf{S5D}^r$-formulas,

- $\mathfrak{M}, w \models [\Phi]_D\, \psi$ iff for all $v \in W$ with $(w, v) \in R_\Phi$, $\mathfrak{M}, v \models \psi$, where $R_\Phi = \bigcap_{\varphi \in \Phi} R_\varphi$.

The following proposition shows how any finite preorder can be represented as an intersection of one-step frames. This hints at the possibility that an arbitrary modal operator can be represented as a distributed knowledge of some hypothetical knowledge operator. Here we assume that knowledge operators are box-modalities for logics in the interval $[S4, S5]$. Below we show a version of this claim.

Proposition 1. *Let $W = \{w_1, \ldots, w_k\}$ be a set. Let R be a preorder over W. For all $i \in \{1, \ldots, k\}$, let $R_i = (W \setminus R(w_i)) \otimes R(w_i)$. Then it holds that $R = \bigcap_{i=1..k} R_i$.* ⊣

Intuitively, the proposition can be understood as follows. Observe that a preorder R induces a partial order (i.e. an antisymmetic preorder) on the set of R-clusters, which are sets of points fully connected by R. In other words, R gives rise to a collection of directed graphs whose nodes are R-clusters. Note that the graph is loopless (and thus antisymmetric). Now, if R is total, all points are connected which gives rise to just one such graph. If, additionally, R is 'one-step', the graph consists of merely two nodes. Intersecting one-step total preorders has the effect of erasing directed edges from the universal relation. Note that the intersection of preorders is again a preorder. Proposition 1 shows that by intersecting a certain selection of one-step total preorders, we can "carve out" the desired preorder. The following example illustrates the scenario.

Example 3. *Let $W = \{x, y, z\}$ be a set and $R = \{(x, y), (x, z)\} \cup \mathit{id}(W)$. It is readily checked that R is a reflexive and transitive relation. Now let $R_w = (W \setminus R(w)) \otimes R(w)$ for all $w \in W$. That is, $R_x = W \times W$, $R_y = \{(x, y), (z, y), (x, z), (z, x)\} \cup \mathit{id}(W)$ and $R_z = \{(y, z), (x, z), (x, y), (y, x)\} \cup \mathit{id}(W)$. Intersecting these relations we obtain $R_x \cap R_y \cap R_z = \{(x, y), (x, z)\} \cup \mathit{id}(W)$, which is equal to R.*

The intersection in Proposition 1 reminds us on the relations R_Φ determined by a finite set Ψ of $\mathsf{S5D}^r$-formulas in a model. In fact, this is the connection we seek to establish in order to represent the knowledge of an agent as distributed knowledge. In the following, we state how this is done.

Take an arbitrary uni-modal logic L between S4 and S5 (whose satisfaction relation is denoted by $\models_{\mathrm{L}}$). The necessity operator '$\Box$' of L is thought of as representing the knowledge of the agent. Note that the system L contains the axioms (T) and $(\mathsf{4})$, each of which represent important epistemic properties, namely, veridicality and positive introspection, respectively. Of course, L may contain other axioms, in fact, any axiom that can be derived in system S5. We assume that L is determined by a class $\mathcal{C}$ of Kripke structures (i.e., the theorems of L are exactly the formulas that are valid on all structures in $\mathcal{C}$). Clearly, the structures in $\mathcal{C}$ are reflexive and transitive. What we require as a precondition is that L has the finite-model property w.r.t. $\mathcal{C}$. This means that, if a formula φ is not a theorem of L then there is a finite Kripke structure $\mathfrak{M}^k$ in $\mathcal{C}$ that falsifies φ, i.e. $\mathfrak{M}^k, w \not\models \varphi$ for some world w in $\mathfrak{M}^k$.

Before we can state the theorem, we need one more auxiliary notion. Let $\mathfrak{M}^k = (W, R, V)$ be a finite Kripke structure such that the relation R is a preorder. We say that the valuation function V *covers* R if for every world $w \in W$, there is an atomic proposition p_w such that $V(p_w) = R(w)$, i.e. the R-image at w.

Theorem 2. *Let $\mathcal{C}$ be a class of Kripke structures whose relations are preorders. Let $\mathfrak{M}^k = (W, R, V)$ be a finite structure from $\mathcal{C}$ such that V covers R. Let $\mathfrak{M} = (W, V)$ be a basic structure and let $w \in W$ be a world. Let φ be a Boolean formula over Π. Then, there is a finite set Ψ of atomic propositions such that the following are equivalent:*

(i) $\mathfrak{M}^k, w \models_L \Box\varphi$;

(ii) $\mathfrak{M}, w \models_{\mathsf{S5D}^r} [\Psi]_D\, \varphi$. $\dashv$

Proof. For every $w \in W$, select an atomic proposition p_w such that $V(p_w) = R(w)$. Note that such p_w exists since V covers R. Set $\Psi = \{ p_w \mid w \in W \}$. Using Lemma 1 the equivalence of (i) and (ii) can be shown by induction on the structure of φ. $\square$

We remark that the theorem can be generalized since the condition of using finite models is a bit too strict. Recall the metaphor that views a preorder R as a collection of loopless graphs whose nodes are R-clusters. What is actually required is that the collection of graphs and the graphs themselves are finite. So, we can still find a finite intersection of relations as desired.

The following example illustrates Theorem 2 and discusses the presented notions.

Example 4. *Consider the Kripke model $\mathfrak{M}^k = (W, R, V)$, where W and R are as in Example 3, and $V(p) = \{x, z\}$ and $V(q) = \{z\}$. Clearly, $\mathfrak{M}^k$ is not an S5-model as R is not symmetric. Let $\varphi_{.2}$, $\varphi_{.3}$, $\varphi_{.4}$ be the instances of the axioms (.2), (.3) and (.4) as shown above. It turns out that only $\varphi_{.3}$ holds at x, but not $\varphi_{.2}$ nor $\varphi_{.4}$. In fact, $\varphi_{.3}$ holds at all worlds in $\mathfrak{M}^k$. Let us assume that the box (i.e., the epistemic capacity of the agent) is characterized by the system S4.3.*

Now label the worlds with fresh atomic propositions p_x, p_y, p_z, i.e., we set $V'(p_w) = \{w\}$ for all $w \in W$. Notice that V' covers R. Let R_{p_x}, R_{p_y}, R_{p_z} be the relations determined by the basic structure $\mathfrak{M} = (W, V')$ and the fresh propositions. Notice that R_{p_w} equals R_w from Example 3, for every $w \in W$. Thus $R_{p_x} \cap R_{p_y} \cap R_{p_z} = R$. Now it is immediate that $\mathfrak{M}, w \models [\{p_x, p_y, p_z\}]_D\, \varphi$ iff $\mathfrak{M}^k, w \models \Box\varphi$, for all $w \in W$ and all propositional formulas φ without occurrence of any of p_x, p_y and p_z. In other words, $[\{p_x, p_y, p_z\}]_D$ simulates the S4.3-box. We can see p_x, p_y, p_z as hypotheses that another agent has to adopt in order to know what the S4.3-agent knows.

In some cases, we have an alternative to introducing fresh propositions even though V does not cover R. This means that V covering R is a sufficient but not necessary condition for Theorem 2. Here $\neg p$ and q are hypotheses so that $[\{\neg p, q\}]_D$ simulates S4.3-box as well. That is, hypotheses do not need to be atomic propositions. Moreover, (parts of) hypotheses may occur in the conclusion as in $(W, V), x \models [\{\neg p, q\}]_D\, \varphi_{.3}$.

The reduction of the distributive knowledge modality for a finite set $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ of agents is recursively defined as a function Rd as follows, where $n \geq 3$:

$$\begin{array}{rcl} \mathsf{Rd}([\varphi_1]_D\, p) & := & [\varphi_1]_K\, p \\ \mathsf{Rd}([\varphi_1, \varphi_2]_D\, p) & := & [\varphi_2]_K\, p \vee (\varphi_1 \wedge [\varphi_2]_K\, (\varphi_1 \to p)) \\ \mathsf{Rd}([\varphi_1, \ldots, \varphi_n]_D\, p) & := & \mathsf{Rd}([\mathsf{Rd}([\varphi_1, \ldots, \varphi_{n-1}]_D\, p), \varphi_n]_D\, p) \end{array}$$

Here we assume that each φ_i belongs to the language of $\mathsf{S5}^r$, i.e., φ_i does not contain the distributive knowledge modality although the reduction for arbitrary formulas $\Phi = \{\varphi_1, \ldots, \varphi_n\}$, where the φ_i-s may include the distributive knowledge modality, is a simple application of the reduction step by step. It can readily be seen that the function Rd yields a formula in the language of $\mathsf{S5}^r$.

Theorem 3. *Let Φ be a finite set of formulas in the language of $\mathsf{S5}^r$. Then the formula $[\Phi]_D\, p \leftrightarrow \mathsf{Rd}([\Phi]_D\, p)$ is a relational modal definition for the language of $\mathsf{S5}^r$.*

Let $\mathsf{S5D}^r$ be the logic obtained by extending $\mathsf{S5}^r$ with modal definitions of the form $[\Phi]_D\, p \leftrightarrow \mathsf{Rd}([\Phi]_D\, p)$, where Φ ranges over sets of formulas in the language of $\mathsf{S5}^r$ and the function Rd is defined as above. We obtain the following result.

Theorem 4. *The modal logic $\mathsf{S5D}^r$ is sound and complete w.r.t. the class of all basic structures.*

The proof of the theorem follows from the fact that the modal logic $\mathsf{S5}^r$ is complete and theorems 1 and 3. We also imply decidability of the logic since all formulas are reduced to $S5^r$ formulas.

Theorem 5. *The logic $\mathsf{S5D}^r$ is decidable.*

Proof. Proof follows from decidability of the logic $S5^r$ and reducibility of an arbitrary $\mathsf{S5D}^r$ formula to a $\mathsf{S5}^r$ formula. □

4 Conclusion

In this paper we studied $\mathsf{S5D}^r$ which is an extension of $\mathsf{S5}^r$ by distributive knowledge operator. In particular we have addressed Kripke completeness and decidability issues. In previous works we have investigated similar questions for $\mathsf{S5}^r$ and $\mathsf{S5C}^r$ ($\mathsf{S5}^r$ extended with common knowledge operator) as well. This paper summarises investigation of Kripke semantics of the logics mentioned above. In future we aim to investigate topological semantics of the logics as well as computational complexity of the satisfiability problem for the logics.

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