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ARCH-COMP22 Category Report: Hybrid Systems Theorem Proving

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Abstract

This paper reports on the Hybrid Systems Theorem Proving (**HSTP**) category in the ARCH-COMP Friendly Competition 2022. The characteristic features of the HSTP category remain as in the previous editions [MST+18, MST+19, MMJ+20], it focuses on flexibility of programming languages as structuring principles for hybrid systems, unambiguity and precision of program semantics, and mathematical rigor of logical reasoning principles. The benchmark set includes nonlinear and parametric continuous and hybrid systems and hybrid games, each in three modes: fully automatic verification, semi-automatic verification from proof hints, proof checking from scripted tactics. This instance of the competition focuses on presenting the differences between the provers on a subset of the benchmark examples.

1 Introduction

This report summarizes the experimental results of the Hybrid Systems Theorem Proving (HSTP) category in the ARCH-COMP22 friendly competition, focusing on a feature comparison between the participating theorem provers. Details on the benchmark sets and the evaluation modes can be found in previous editions of the HSTP category [MST+18, MST+19, MMJ+20]. The 218 examples in the benchmark competition are grouped into the following categories:

- Hybrid systems design shapes: small-scale examples over a large variety of model shapes to test for prover flexibility.
- Nonlinear continuous models: test for prover flexibility in terms of generating and proving properties about continuous dynamics, based on [SMT+19, SMT+20].

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- Hybrid games: small-scale examples with adversary dynamics in differential dynamic game logic.
- Hybrid systems case studies: hybrid systems models and specifications at scale to test for application scalability and efficiency, based on [MGVP17].
- Hybrid systems from Simulink/Stateflow models: examples translated from Simulink/Stateflow models to verify.

In each of these categories, tools can select the degree of automation depending on their focus in the spectrum from fast proof checking to full proof automation:

(A) Automated: hybrid systems models and specifications are the only input, proofs and counterexamples are produced fully automatically.

(H) Hints: select proof hints (e.g., loop invariants) are provided as part of the specifications.

(S) Scripted: significant parts of the verification is done with dedicated problem-specific scripts or tactics.

Benchmark examples in the hybrid systems design shapes, nonlinear continuous models, hybrid games and hybrid systems case study benchmarks are available at https://github.com/LS-Lab/KeYmaeraX-projects/tree/master/benchmarks and specified in differential dynamic logic (dL) [Pla08, Pla17]. Benchmark examples for HHLPy, including the Simulink/Stateflow models and their translations to Hybrid CSP [ZWR95], are available at https://gitee.com/bhzhan/mars/tree/master/hhlpy/examples/simulink. The participating tools are presented in Section 3. An overview of the examples together with the findings from the competition is given in Section 4.

2 Problem Format

Benchmarks in the hybrid systems design shapes, nonlinear continuous models, hybrid games and hybrid systems case study benchmarks categories are written in differential dynamic logic (dL) [Pla08, Pla17] which has axioms and an unambiguous semantics available [BRV+17] in KeYmaera 3, KeYmaera X, Isabelle/HOL, and Coq. A tutorial on the modeling principles in dL can be found in [QML+16], details on the ASCII syntax are in [MMJ+20]. Here, we list the extensions over [FMBP17] to the scripting language that are introduced instances of the competition. Benchmarks in the hybrid systems design shapes and nonlinear continuous models are also translated to the HHLPy input language, along with the Simulink/Stateflow benchmarks. In the second subsection, we describe the input language for HHLPy in the competition.

Scripting Language ASCII syntax. The KeYmaera X ASCII syntax is illustrated in the example below, with tactics using position identifiers to refer to formulas and terms in a sequent.

The proof script language uses search locators, e.g., `implyR('R==...)` in line 2 refers to a formula in the alternatives to prove (right-hand side of the sequent turnstile), as opposed to `implyR(1)`, which refers to a fixed position in the sequent and is vulnerable to changes in formula ordering in case automated tactics progress in proofs differently across prover versions. Tactics can use marker `#` to refer to sub-formulas or terms: e.g., the locator `simplify('R=="x>=0 & #y+0#>=x")` applies tactic `simplify` to term `y+0`. Branch labels (e.g., `"Init"` in line 4) unambiguously identify on which of the branches to apply some tactic. The branch labels themselves are created by tactics, e.g., tactic `loop` generate labels `Init`, `Post`, and `Step` in lines 4, 6, and 8, respectively, while tactic `andR` generates labels according to the formula it was applied to.

```
1 ArchiveEntry "Benchmark Example 1"
2
```

```
Definitions                                  /* definitions cannot change their value */
  Real A = 5;                                /* real−valued maximum acceleration defined to be 5 */
  Real b;                                    /* real−valued braking, undefined so unknown value */
  Bool geq(Real x, Real y) <−> x>=y;         /* predicate geq defined to be formula x>=y */
  HP drive ::= {                             /* program drive defined to choose either */
        ?v<=5; a:=A;                         /* maximum acceleration if slow enough */
    ++ a:=−b;                                /* or braking, nondeterministically */
  };
End.

ProgramVariables                 /* program variables may change their value over time */
  Real x;                        /* real−valued position */
  Real v;                        /* real−valued velocity */
  Real a;                        /* current acceleration chosen by controller */
End.

Problem                          /* conjecture in differential dynamic logic */
  v>=0 & b>0                     /* initial condition */
  −>                             /* implies */
  [                              /* all runs of this hybrid program */
    {                            /* braces {} group programs */
      drive;                     /* expand program drive here as defined above */
      { x'=v, v'=a & v>=0 }      /* differential equation system */
    }* @invariant(v>=0)          /* loop repeats, with @invariant contract */
  ] v>=0                         /* safety/postcondition after hybrid program */
End.

Tactic "Automated proof in KeYmaera X"
  auto
End.

Tactic "Scripted proof in extended Bellerophon tactic language"
implyR('R=="v>=0&b()>0−>[{{?v<=5;a:=5;++a:=−b();}{x'=v,v'=a&v>=0}}*]v>=0");
loop("v>=0", 'R=="[{{?v<=5;a:=5;++a:=−b();}{x'=v,v'=a&v>=0}}*]v>=0"); <( /* < splits branches */
  "Init":
    id,       /* initial case: shown with close by identity */
  "Post":
    QE,       /* postcondition: prove by real arithmetic QE */
  "Step":
    composeb('R=="[{?v<=5;a:=5;++a:=−b();}{x'=v,v'=a&v>=0}]v>=0");
    solve('R=="[?v<=5;a:=5;++a:=−b();]#[{x'=v,v'=a&v>=0}]v>=0#");
    choiceb('R=="[?v<=5;a:=5;++a:=−b();]\forall t_ (t_>=0−>\forall s_ (0<=s_&s_<=t_−>a*s_+v
        ↪ >=0)−>a*t_+v>=0)");
    /* separate controller branches */
    andR('R=="[?v<=5;a:=5;]\forall t_ (t_>=0−>\forall s_ (0<=s_&s_<=t_−>a*s_+v>=0)−>a*t_+v
        ↪ >=0)&[a:=−b();]\forall t_ (t_>=0−>\forall s_ (0<=s_&s_<=t_−>a*s_+v>=0)−>a*t_+v
        ↪ >=0)"); <(
      "[?v<=5;a:=5;]\forall t_ (t_>=0−>\forall s_ (0<=s_&s_<=t_−>a*s_+v>=0)−>a*t_+v>=0)":
        /* decompose some steps then ask auto */
        composeb('R=="[?v<=5;a:=5;]\forall t_ (t_>=0−>\forall s_ (0<=s_&s_<=t_−>a*s_+v>=0)−>
            ↪ a*t_+v>=0)");
        testb('R=="[?v<=5;][a:=5;]\forall t_ (t_>=0−>\forall s_ (0<=s_&s_<=t_−>a*s_+v>=0)−>a*
            ↪ t_+v>=0)");
        auto,
      "[a:=−b();]\forall t_ (t_>=0−>\forall s_ (0<=s_&s_<=t_−>a*s_+v>=0)−>a*t_+v>=0)":
        /* assignment, then real arithmetic */
        assignb('R=="[a:=−b();]\forall t_ (t_>=0−>\forall s_ (0<=s_&s_<=t_−>a*s_+v>=0)−>a*t_
            ↪ +v>=0)");
        QE
    )
)
End.

End. /* end of ArchiveEntry */
```

Proof scripts expressed in the locator style `'R==...` and `'L==...` are more explicit about how they operate on specific examples, but such scripts do not transfer well between different examples. In order to generalize a script for applicability to other examples, or to increase robustness to changes in the input model, locators `'R~=...` and `'L~=...` use unification to decide where to apply tactics.

Input Language for HHLPy Benchmarks in the Hybrid Systems from Simulink/Stateflow category are modeled using Hybrid CSP, with properties specified as Hoare triples in Hybrid Hoare Logic. Both Hybrid CSP programs and properties as Hoare triples can be written using an ASCII syntax, as illustrated below. They are composed of pre-condition, program and post-condition. The program is annotated with invariants and rules for proof.

```
# ArchiveEntry Benchmark Example 2

pre;                                          # pre-conditon
t := 0;                                       # Assignment command
x := 0;
{                                             # braces {} group programs
  Chart_A_done := 0;
  if (t >= 1) {                               # If command
    t := 0;
    x := 0;
    Chart_A_done := 1;
  }
  Chart_ret := Chart_A_done;
  {x_dot = 1, t_dot = 1 & t < 1}              # differential equation systems
    invariant [x == t]{di};                   # differential equation invariant, with proof rule
}*                                            # loop repeats
  invariant [x == t] [0 <= x] [x <= 1];       # loop invariant
post [0 <= x] [x <= 1];                       # post-condition
```

The pre-condition is true and the post-condition is $0 \leq x \wedge x \leq 1$ in the above example. In the program, t and x are assigned as 0, followed by a loop command. The invariants of the loop are $x == t$, $0 \leq x$ and $x \leq 1$. A differential equation is in the loop with invariant $x == t$ proved by the rule dI (differential invariant).

3 Participating Tools

KeYmaera X. KeYmaera X [FMQ+15] is a theorem prover for the hybrid systems logic differential dynamic logic (dL). It implements the uniform substitution calculus of dL [Pla17]. A comparison of the internal reasoning principles in the KeYmaera family of provers with a discussion of their relative benefits and drawbacks is in [MP20], and model structuring and proof management on top of uniform substitution is discussed in [Mit21]. KeYmaera X supports systems with nondeterministic discrete jumps, nonlinear differential equations, nondeterministic inputs, and allows defining functions implicitly through their characterizing differential equations [GTMP22]. It provides invariant construction and proving techniques for differential equations [SMT+21, PT20], and stability verification techniques for switched systems [TMP22]. To discharge proof obligations in real arithmetic, KeYmaera X interacts with trusted backend procedures for quantifier elimination (Z3, Wolfram Mathematica, Wolfram Engine); a verified backend procedure based on virtual term substitution is under development [SCMP21]. Proofs in KeYmaera X can be conducted interactively [MP16a], steered with tactics [FMBP17], or attempted fully automatic. Compared to previous editions of this benchmark, KeYmaera X development focused on extending interactive and scripted proof capabilities (for switched systems and user-defined functions), and unified several automation heuristics into easier-to-maintain and more broadly applicable automated tactics.

HHL Prover/HHLPy. HHL Prover is a verification tool for hybrid systems modelled by Hybrid CSP (HCSP) [He94, ZWR96], implemented in Isabelle/HOL. HCSP is an extension of CSP by introducing differential equations for modeling continuous evolution and interrupts for modeling interaction between continuous and discrete dynamics. The proof system of HHL Prover is Hybrid Hoare Logic (HHL) [LLQ+10].

HHLPy is a new verification tool for HCSP, that provides a friendlier web-based user interface. Currently it handles only the sequential fragment of HCSP, with reasoning rules similar to those in dℒ. We briefly introduce each of the two tools in the following paragraphs.

HHL Prover HHL Prover [WZZ15] is an interactive theorem prover for verifying hybrid systems modelled by Hybrid CSP (HCSP). We use the trace-based hybrid Hoare logic for reasoning about HCSP processes as in last year. Traces for both sequential and parallel HCSP processes are represented as lists of *trace blocks*. There are two types of trace blocks: ODE blocks and communication blocks. ODE blocks specify evolution of the process over an interval of time, consisting of duration of the interval, the state of the process as a function of time, and a set of communications that are ready during the interval. Communication blocks are of three types: input, output, and IO. Input and output blocks specify an unmatched communication event, while IO blocks specify a matched communication event. All three types of events also specify the value that is communicated.

HHLPy HHLPy is a theorem prover with a friendlier user interface, currently for verifying sequential HCSP programs only. It is implemented using Python and JavaScript. The sequential fragment of HCSP contains ODEs with domain boundary, but not communication, interrupts, and parallel processes. Extending HHLPy to handle the full HCSP language is left for future work. Given a sequential HCSP process P, a specification takes the form of Hoare triple, $\{Pre\}P\{Post\}$, where Pre and $Post$ are pre-/post-conditions in first-order logic.

To reason about differential equations, HHLPy makes use of a set of proof rules that are inspired by dℒ [Pla10, Pla11, PT18], but adapted to the semantics of sequential HCSP. The differential weakening (dW) rule reduces a Hoare triple concerning ODEs to an invariant triple of the ODE and some verification conditions. Invariant triple is in the form of $[\![P]\!]\langle \dot{x} = e\rangle[\![Q]\!]$, whose semantics is roughly stated as follows: for any solution to the differential equation $\dot{x} = e$, if Q is satisfied at beginning and P is satisfied throughout, then Q is satisfied throughout. Rules such as differential invariant (dI), differential cut (dC), Darboux's rule (dbx) and barrier certificates (barrier), many of which borrowed from differential dynamic logic, are then used to prove invariant triples.

HHLPy stores proof information after the corresponding assertion (post-condition, invariant), so that the user can still reuse proofs when they modify the program or the assertions slightly. Specifically, proof rules are stored after the corresponding invariants, and proof methods for proving verification conditions, for example, Z3 or Wolfram Engine, are stored after the assertion that generates the corresponding verification condition. Sometimes one assertion corresponds to several verification conditions. HHLPy also proposed a labeling system to distinguish these verification conditions.

IsaVODEs. **Isa**belle **V**erification with **O**rdinary **D**ifferential **E**quation**s** [FHGS21] is an extension of a hybrid systems verification component [HS22, Hue19] with lenses from the Unifying Theories of Programming (UTP) framework [HH98, FBC+20]. IsaVODEs shallowly represents programs as state transformers and propositions as predicates, while lenses algebraically characterise access and mutation of state. Proving soundness in Isabelle/HOL of Hoare logic and weakest liberal precondition (wlp) laws makes them available for verification with IsaVODEs. In particular, IsaVODEs provides its own wlp-rules for reasoning about hybrid programs. It also offers dℒ-like syntax and all the major dℒ-rules. Specifically, IsaVODEs now includes rules for reasoning with forward diamonds and it can manipulate differential invariants with cuts, ghosts, and inductions in the style of dℒ. Since IsaVODEs is only restricted to Isabelle's higher

order logic (HOL), it also offers support for direct reasoning with transcendental functions like exponentials, sines and cosines. Similarly, it can encode linear systems of ODEs as operations between matrices and vectors [Hue20a, Hue20b]. IsaVODEs also provides more automation than its predecessor verification components [HS22, FHS20]. It has Eisbach-methods for automatically discharging common differential induction and Hoare logic arguments, and it adds support for manipulating real arithmetic expressions. Through the Wolfram engine, IsaVODEs can suggest solutions to systems of ODEs to aid in the verification process. We use Isabelle's pdf-generation tool to produce a proof-document with our solutions to the competition's problems in our reproducibility package. Recent IsaVODEs developments are available online[1].

4 Benchmarks

One of the strengths of hybrid systems theorem proving as a verification technique is its support for combined automated and interactive verification steps as well as its applicability to proof search and proof checking. The benchmark examples were analyzed in three modes:

Automated The specification is the only input to the theorem prover. Proofs and counterexamples are obtained fully automated to highlight the capabilities of theorem provers in terms of invariant generation, proof search, and proof checking.

Hints Known design properties of the system, such as loop invariants and invariants of differential equations, are annotated in the model and allowed to be exploited during an otherwise fully automated proof to highlight the capabilities of theorem provers in terms of proof search and proof checking.

Scripted User guidance with proof scripts is allowed to highlight the capabilities of theorem provers in terms of proof checking.

The benchmark examples are structured into 5 categories: hybrid systems design shape examples to test for system design variations at a small scale, nonlinear continuous models to test for continuous invariant construction and proving capabilities, hybrid game examples to test adversarial dynamics, hybrid systems case studies to test for prover scalability, and a new category for hybrid systems from Simulink/Stateflow models.

Experimental setup. IsaVODEs participated only in scripted format in the hybrid systems design shapes problems and in the European train control System case study benchmark. The IsaVODEs setup was an 8 core Apple M1 machine with a 16 GB memory.

HHLPy participated in three benchmark sets, which are hybrid systems design shapes, nonlinear continuous models and hybrid systems from Simulink/Stateflow models. The performance results were obtained on Windows11, with Z3-solver 4.8.12.0 and Wolfram Engine 13.0, on the same machine with 8-core Intel(R) Core(TM) i5-1035G4 CPU @ 1.10GHz and 16 GB memory.

HHL Prover participated in hybrid systems design shapes. The results were obtained on Windows10, with Isabelle2020 and afp-2020-12-22 on the machine with Intel(R) Core(TM) i7-8550U CPU @ 1.80GHz.

KeYmaera X participated in scripted, hints, and automated mode in the design shapes, games, nonlinear continuous models, and case study benchmarks categories. The performance results reported here are obtained on MacOS, with KeYmaera X 5.0, Wolfram Mathematica 12.3, and Pegasus invariant generator on Matlab 2021b with SOSTools 3.04. The proof duration

[1] https://github.com/isabelle-utp/Hybrid-Verification

measurements were taken on a 2013 Mac Pro with 6-core Intel Xeon E5 3.5GHz and 28GB memory; KeYmaera X uses a single core, but hands some real arithmetic proof obligations to Mathematica for parallel execution inside Mathematica.

4.1 Hybrid Systems Design Shapes

This category is designed to test for basic verification features on simple examples. The benchmark examples are grouped as follows:

Static semantics correctness 9 examples with various sequential orders and nested structures of assignments, differential equations, and loops.

Dynamics 30 examples with differential equations ranging from solvable to nonlinear.

LICS Tutorial 9 dℒ tutorial examples [Pla12] ranging from basic time-triggered motion control to model-predictive control.

STTT Tutorial 12 dℒ modeling tutorial examples [QML+16] ranging from basic discrete event-triggered and time-triggered control for straight-line motion to speed control with a trajectory generator and lane-keeping with two-dimensional curved motion.

Harmonic Oscillator This year, we add a new design shape problem, see Section 5.1.

KeYmaera X KeYmaera X participated in the scripted, hints, and automated format of the competition (proof attempts were aborted after 60s, every proof attempt was made in a fresh prover instance with all caches cleared): In the scripted format, KeYmaera X solved 60 of the 61 examples with a total amount of 119s for checking the scripted proofs. The remaining unsolved example is "LICS: Example 4b progress of time-triggered car", where the script advances the proof using a parametric variant [Mit21] but misses a witness for its existential quantifier. In the hints format, KeYmaera X solved 52 of the 61 examples with a total amount of 109s for filling in the missing proof details and checking the proofs. In the automated format, KeYmaera X generated proofs for 51 of the examples, with a total duration of 144s. Compared to previous instances of this competition [MJZ+21, MMJ+20], automation solves a few examples less, which is likely caused by hybrid systems and ODE automation refactoring to a more general (but not yet complete) implementation with less specialized heuristics.

HHL Prover/HHLPy. The HHL Prover successfully proved 49 of the 61 examples in Isabelle/HOL using our proof system. Since the benchmarks are originally formulated in terms of dynamic logic, some modifications are made to adapt it to a Hoare-logic style system.

HHLPy successfully solved 50 of the 61 design shapes problems. 8 of the unsolved problems are hard to translate into a Hoare-logic style system or a HCSP program. Another unsolved problem is "Dynamics: Exponential growth (4)", which is with a non-polynomial expression that HHLPy cannot handle yet. For another two unsolved problems, "STTT Tutorial: Example 10" and "Harmonic Oscillator", we have not found appropriate invariants or rules to prove them. All 50 problems were verified in 12.84 seconds, which means 0.26 seconds per problem.

IsaVODEs. The IsaVODEs' team participated only in the scripted format of the competition. We solved 59 of the 61 Design Shapes problems with it. We fully proved 55 of those 59, while the remaining 4 required involvement of the WolframEngine. Computer algebra systems usually help interactive provers discharge proof obligations about real-arithmetic. In our case, the WolframEngine reduced useful (in)equalities to the boolean value `true`. Thus, we could assert (but not prove) within Isabelle those (in)equalities and use them as facts in our proofs of the

aforementioned four Design Shapes. Isabelle-asserted lemmas appear in our reproducibility package's proof document with `sorry` commands below them.

Due to the competition time constraints, we could not find an argument for the truth of one of the remaining two problems: "LICS: Example 4b progress of time-triggered car". However, we were able to state the verification formula in IsaVODEs and formalise forward diamond operator rules to tackle it. We marked unsolved problems with `oops` in our proof document. The last unsolved problem, "Dynamics: Fractional Darboux equality", requires an unproved variant of Isabelle's Picard-Lindelöef theorem or a generalisation of our differential Ghost rule.

We measured Isabelle's speed to certify our proofs in two ways. Firstly, we measured the time it took Isabelle to certify our proofs for each problem. The average per Design Shape solved problem was 2.39 seconds in our 8 core Apple M1 machine. Alternatively, we measured the time it took the same machine to certify all problems. The computer averaged 34.21 seconds for the first 58 problems, making each of them solved in less than 0.59 seconds. We attribute the disparity between both methods to the inaccuracies generated by manual measurements in the first method and the extra effort the machine needs to display each problem to the monitor.

4.2 Nonlinear Continuous Models

The examples in this category remained unchanged from [MMJ+20] for direct comparison of the verification performance with previous results; the examples test for pure continuous verification performance. Future competitions may additionally utilize the extended benchmark set of [SMT+20].

KeYmaera X. KeYmaera X participated in the scripted, hints, and automated format (proof attempts were aborted after 60s, every proof attempt was made in a fresh prover instance with all caches cleared): In the scripted format, KeYmaera X solved 105 of the 141 examples in 247s, with hints 94 examples were solved in 585s, and fully automatic with invariant generation and checking, KeYmaera X solved 63 examples in 1344s. Compared to previous instances of this competition [MJZ+21, MMJ+20], automation solves considerably less examples and points to potential regression in both the differential invariant generator and the invariant checking tactics due to partial re-implementation.

HHLPy. HHLPy successfully solved 90 of the 141 nonlinear continuous model problems. Proof attempts were aborted after a timeout of 300 seconds. For most of the unsolved problems, we have not come up with appropriate invariants or proof rules. Some other remaining ones are too slow to give a result. All 90 problems were verified in 48.67 seconds, which means 0.54 seconds per problem. The longest successful verification among all problems took 35.27 seconds alone.

4.3 Hybrid Games

The hybrid games benchmark tests basic games reasoning over 3 examples with adversarial dynamics. Future editions of the competition may utilize extended games case studies, such as [CMP23].

KeYmaera X. KeYmaera X solves all 3 examples in scripted mode, and 2 examples from hints and fully automatic, with an average duration of about 1s per example in each format. Again proof attempts were aborted after 60s, every proof attempt was made in a fresh prover instance with all caches cleared.

4.4 Hybrid Systems Case Study Benchmarks

Category overview. The benchmark examples in this category are selected to test theorem provers for scalability and efficiency on examples of a significant size and interest in applications and remained unchanged from [MST+19]. The benchmark examples[2] are inspired from prior case studies on train control [PQ09, ZLW+13], flight collision avoidance [PC09], robot collision avoidance [MGVP17], a lunar lander descent guidance protocol [ZYZ+14], and rollercoaster safety [BLCP18].

KeYmaera X. KeYmaera X participated in the scripted, hints, and automated format (proof attempts were aborted after 300s, every proof attempt was made in a fresh prover instance with all caches cleared), and attempted 8 examples (3 ETCS train control, 3 flight collision avoidance, 2 robot collision avoidance). In the scripted format, KeYmaera X solved all 8 attempted examples with a total time of $225s$; in the hints format 7 examples in $203s$ (robot collision avoidance passive safety timed out in a real arithmetic proof obligation in the backend real arithmetic solver); and in fully automated mode, KeYmaera X solved 5 examples in $270s$. The remaining examples not solved automatically include the 4-aircraft flight collision avoidance and both robot collision avoidance examples (these examples feature high arithmetic complexity and time out in the backend real arithmetic solvers). Automation in the case study benchmarks subcategory remained at the level of previous instances of this competition [MJZ+21, MMJ+20].

IsaVODEs. Our participation was in the scripted format of the competition. We tackled the European train control system (ETCS) case study benchmark. ETCS is further divided in three: essentials (safety), controllability and reactivity. We fully verified the essentials part of the problem with IsaVODEs. With the help of the WolframEngine, we also solved the controllability problem. We were able to formalise ETCS reactivity in Isabelle, but we did not find an argument for the truth of the problem within the timespan of the competition.

4.5 Hybrid Systems from Simulink/Stateflow Models

Category overview. This category contains hybrid systems modeled using Simulink/Stateflow. These models are first translated into the modeling language used for verification, and then its properties are verified using the appropriate tools. As this is the first year of competition including this category, only 4 benchmark problems are included, which illustrates the basic semantics of Simulink and Stateflow[3]. They include Stateflow charts with one or two states, ODEs within each state, and delay blocks in Simulink diagrams.

HHLPy. HHLPy successfully solved all 4 problems. They were verified in 1.61 seconds, which means 0.40 seconds per problem.

5 Modeling and Proof Comparison

5.1 Harmonic oscillator

The second-order ODE $x''(t) = a \cdot x(t) + b \cdot x'(t)$ represents a harmonic oscillator. It can be encoded as a linear system of ODEs and embedded in a hybrid program [Hue20a] (example

[2] https://github.com/LS-Lab/KeYmaeraX-projects/blob/master/benchmarks/advanced.kyx
[3] https://gitee.com/bhzhan/mars/tree/master/hhlpy/examples/simulink

6.1). Thus, this year's competition now includes the dℒ-hybrid program below.

```
Problem
  b^2+a*4 > 0 & a<0 & b<=0
  ->
  [ {
      x:=*; ?x>0; y:=0;
      {x'=y, y'=a*x+b*y}
    }*@invariant(x>=0)
  ]x>=0
End.
```

The term $b^2 + 4 \cdot a$ is the oscillator's damping factor [Att03]. The hybrid program then states that releasing the oscillator starting from rest ($y = 0$) and arbitrarily extended ($x > 0$) will keep the oscillator extended in an overdamped system ($b^2 + a * 4 > 0$, $a < 0$ and $b <= 0$).

KeYmaera X The KeYmaera X model below expresses the response of the oscillator to a non-deterministic input x with a differential equation of non-deterministic duration (any non-negative duration). Inputs and response can be repeated any non-deterministically chosen number of times with the loop. The proof shows that a loop invariant $x \geq 0$ is maintained, because the differential equation maintains the invariant $-\frac{-b+\sqrt{b^2+4a}}{2}x \leq y \wedge y \leq 0$, which implies $x \geq 0$. The proof shows $x \geq 0$ for any number of repetitions and any combination of inputs $x > 0$ and durations (infinitely many combinations and durations in each of the infinitely many loop repetitions).

```
Theorem "Benchmarks/Basic/Affine: Overdamped Door Closing Mechanism"

Definitions
  Real a, b;
End.

ProgramVariables
  Real x, y;
End.

Problem
  x=0 & b^2+a*4 > 0 & a<0 & b<=0 ->
  [{x:=*; ?x>0; y:=0;
    {x'=y, y'=a*x+b*y}
   }*@invariant(x>=0)
  ]x>=0
End.

Tactic "Scripted proof"
implyR(1);
loop("x>=0", 'R=="[{x:=*;?x>0;y:=0;{x'=y,y'=a*x+b*y}}*]x>=0"); <(
  "Init": QE using "x=0 :: x>=0 :: nil",
  "Post": id,
  "Step":
    unfold;
    cut("\exists w w=(-b+(b^2+4*a)^(1/2))/2"); <(
      "Use":
        existsL('L=="\exists w w=(-b+(b^2+4*a)^(1/2))/2");
        dC("-w*x<=y&y<=0", 'R=="[{x'=y,y'=a*x+b*y}]x>=0"); <(
          "Use":
            dW('R=="[{x'=y,y'=a*x+b*y&true&-w*x<=y&y<=0}]x>=0");
            QE,
          "Show":
            ODEinv('R=="[{x'=y,y'=a*x+b*y}](-w*x<=y&y<=0)")
        ),
      "Show":
        QE using "b^2+a*4>0 :: \exists w w=(-b+(b^2+4*a)^(1/2))/2 :: nil"
    )
)
End.
```

The induction proof in Line 22–42 has 3 subgoals: the loop invariant is true initially (Line 24: $\models x = 0 \to x \geq 0$ by real arithmetic `QE`), it implies the postcondition (Line 26: $\models x \geq 0 \to x \geq 0$ by propositional tautology `id`), and it is preserved through one iteration of the loop body (Line 28–41: $\models x \geq 0 \to [x := *; ?x > 0; y := 0; \{x' = y, y' = ax + by\}]x \geq 0$ by symbolic program execution `unfold` and differential cut `dC` $-wx \leq y \leq 0$ with $w = \frac{-b+\sqrt{b^2+4a}}{2}$, justified by differential invariant reasoning `ODEinv` [PT20]).

HHLPy The "Harmonic Oscillator" benchmark has not been verified in HHLPy yet. Below we only list the modeling of this benchmark in HHLPy. The specification below expresses that the oscillator system always satisfy the postcondition $x \geq 0$ in an overdamped system ($b^2+a*4 > 0$, $a < 0$ and $b \leq 0$ in the pre-condition). The loop states repetition for any number of times. The HCSP program inside the loop states the oscillator is arbitrarily extended ($x := *(x > 0)$) and at rest ($y := 0$) at the beginning, then it is released and evolves according to the differential equation for any non-negative duration.

The semantics of differential equation in HCSP is a little different from that in dℒ. In HCSP, the evolution stops exactly when the domain constraint does not hold, instead of stopping at an arbitrary time inside the domain. Therefore, we introduce a time variable (t) with a non-deterministic non-negative initial value and a negative derivative ($t\_dot = -1$) to model the oscillator that will stop at an arbitrary time. Specifically, the evolution stops when t decreases from its non-deterministic non-negative initial value to 0.

```
1 pre [b^2 + a * 4 > 0] [a < 0] [b <= 0];
2 {
3     x := *(x > 0);     # Assign x an arbitrary number that is larger than 0
4     y := 0;
5     t := *(t >= 0);   # Time variable
6     {x_dot = y, y_dot = a * x + b * y, t_dot = -1 & t > 0} # Stop when t > 0 does not hold
7 }*
8 post [x >= 0];
```

IsaVODEs The "Harmonic Oscillator" benchmark has been verified before with the predecessor verification components [Hue20a]. We formalised the dℒ variant for this competition and proved it with IsaVODEs. We showed the equivalence with our previous formalisation (in terms of matrices) and used our results about linear systems of ODEs to verify the specification. Thus, we represented the system of ODEs via $z' = A \cdot z$ where $z = (x, y)^\top$ and

$$A = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}.$$

```
(in Isabelle) abbreviation A ≡ mtx
  ([0, 1] #
   [a, b] # [])
```

In the code above, *mtx* transforms Isabelle lists to square matrices. Then we showed that A is diagonalizable: $A = P \cdot D \cdot P^{-1}$ for a change of basis matrix P and diagonal matrix D.

lemma *mtx-hosc-diagonalizable*:
assumes $b^2 + a * 4 > 0$ **and** $a \neq 0$
shows $A = P\ (-\iota_2/a)\ (-\iota_1/a) * (\text{diag}\ i.\ \textit{if}\ i = 1\ \textit{then}\ \iota_1\ \textit{else}\ \iota_2) * (P\ (-\iota_2/a)\ (-\iota_1/a))^{-1}$
<proof>

We omit the definition of P and abbreviate the 7-line proof for the above lemma with the indicator `<proof>` (see proof document for full proofs). Also, ι_1 and ι_2 are A's eigenvalues in the diagonal matrix D. Therefore, the solution Φ^* to the linear system of ODEs is $\Phi^*(t) = P \cdot \exp(-tD) \cdot P^{-1}$, where exp is the matrix exponential.

```
definition discr ≡ sqrt (b^2 + 4 * a)

abbreviation Φ t ≡ mtx (
  [ι2*exp(t*ι1) − ι1*exp(t*ι2),      exp(t*ι2)−exp(t*ι1)]#
  [a*exp(t*ι2) − a*exp(t*ι1), ι2*exp(t*ι2)−ι1*exp(t*ι1)]#[])

lemma mtx-hosc-solution-eq:
  assumes b^2 + a * 4 > 0 and a ≠ 0
  shows P (−ι2/a) (−ι1/a) * (diag i. exp (t * (if i=1 then ι1 else ι2))) * (P (−ι2/a) (−ι1/a))^−1
  = (1/discr) *R (Φ t)
<proof>
```

Here, $\Phi^*(t) = (1/discr)\Phi(t)$. The fact that this function is the unique solution to the system of ODES corresponds to the Isabelle statement below.

```
lemma local-flow-mtx-hosc:
  assumes b^2 + a * 4 > 0 and a ≠ 0
  shows local-flow ((*V) A) UNIV UNIV (λt. (*V) ((1/discr) *R Φ t))
<proof>
```

We used this solution to prove the corresponding solution for the dℒ-style system of ODEs. Then, we apply usual wlp-reasoning to prove the benchmark's specification.

```
lemma local-flow-hosc: a ≠ 0 ⟹ b^2 + 4 * a > 0
  ⟹ local-flow-on [x ⇝ y, y ⇝ a * x + b * y] (x +L y) UNIV UNIV
  (λt. [x ⇝ x-sol t x y, y ⇝ y-sol t x y])
<proof>

lemma a < 0 ⟹ b ≤ 0 ⟹ b^2 + 4 * a > 0
  ⟹ {x=0}
  LOOP (
    (x ::= ?);(y ::= 0); ¿x>0?;
    {x‘ = y, y‘ = a * x + b * y}
  ) INV (x≥0)
  {x≥0}
  apply (rule hoare-loopI)       (* apply Hoare-rule for loops with invariants *)
    prefer 3 apply expr-simp     (* discharge invariant implies postcondition *)
   prefer 2 apply expr-simp      (* discharge precondition implies invariant *)
  apply (clarsimp simp add:
     wp fbox-g-dL-easiest[OF local-flow-hosc])  (* apply wlp-rules including ODEs solution*)
  apply expr-simp
  apply (clarsimp simp: iota1-def iota2-def discr-def)
  using overdamped-door-arith[of b a] by force  (* discharge remaining real-arithmetic obligations *)
```

5.2 Train Collision Avoidance

KeYmaera X The formalization of the ETCS example [PQ09] uses the definitions mechanism of KeYmaera X [Mit21] to characterize functions and relevant properties of the system, such as initial conditions and a loop invariant which the proof will show to be inductive. The definitions also provide abbreviations for the control (`ctrl`) and dynamics (`drive`) of the train.

```
Definitions
  Real ep; /* Control cycle duration upper bound */
  Real b;  /* Braking force */
  Real A;  /* Maximum acceleration */
  Real m; /* End of movement authority (train must not drive past m) */

  Real stopDist(Real v) = v^2/(2*b);        /* Train stopping distance from speed v with braking force b */
  Real accCompensation(Real v) = ((A/b + 1)*((A/2)*ep^2 + ep*v)); /* Dist. compensate speed increase */
  Real SB(Real v) = stopDist(v) + accCompensation(v); /* Dist. to stop safely when accelerating once */

  /* Initial states */
  Bool initial(Real m, Real z, Real v) <-> (
    v >= 0 &
    m-z >= stopDist(v) & /* train distance to the end of the movement authority sufficient to stop safely */
    b>0 &                                                                    /* brakes are working */
    A>=0 &                                                                   /* engine is working */
    ep>=0
  );

  /* Loop invariant: always maintain sufficient stopping distance */
  Bool loopInv(Real m, Real z, Real v) <-> (v >= 0 & m-z >= stopDist(v));

  HP ctrl ::= {
        ?m - z <= SB(v); a := -b;       /* emergency brake when close to end of movement authority */
    ++ ?m - z >= SB(v); a := A;         /* free driving: accelerate when sufficient distance */
  };

  HP drive ::= {
    t := 0;                                          /* reset control cycle timer */
    {z'=v, v'=a, t'=1  & v >= 0 & t <= ep}    /* drive (not backwards v>=0)
                                                        for at most ep time (t<=ep) until next controller run */
  };
End.
```

The safety specification below lists the (discretely and/or continuously) changing variables of the system and the dL formula of the form $initial \rightarrow [\{ctrl; drive\}^*]safe$.

```
ProgramVariables
  Real a;  /* Actual acceleration -b <= a <= A */
  Real v;  /* Current velocity */
  Real z;  /* Train position */
  Real t;  /* Actual control cycle duration t <= ep */
End.

/* Safety specification of the form: initial -> [{ctrl;plant}*]safe
 * Starting in any state where initial is true,
 * any number of repetitions of running a controller 'ctrl' and then driving according to 'plant'
 * keeps the system safe (end up only in states where 'safe' is true). */
Problem
 initial(m, z, v)  ->
    [
      {
        ctrl;
        drive;
      }*@invariant(loopInv(m, z, v))    /* repeat, loop invariant documents system design property */
    ] (z <= m)                  /* safety property: train never drives past end of movement authority */
End.
```

The proof script below shows that the train maintains sufficient stopping distance, which is a straightforward consequence of the control decisions since the differential equation has a polynomial solution. After `solve` and unfolding program definitions `unfold`, only real arithmetic proof obligations remain, which are each justified by real arithmetic `QE`.

```
implyR(1);
loop("v>=0&m-z>=stopDist(v)", 1) ; <(
"Init": prop,
"Post": QE,
"Step":
  composeb(1);
```

```
  composeb(1.1);
  solve (1.1.1) ;
  unfold;
  doall(QE)
)
```

The proof script language of KeYmaera X lists the justifying proof steps, but does not repeat intermediate proof obligations (lemma statements). If necessary, `print` statements can be inserted into the script to display the proof obligations on the command line, or alternatively, the proof script can be loaded in the KeYmaera X web UI [MP16b] to browse the entire proof in a step-by-step fashion.

HHLPy The HHLPy model of the ETCS example, annotated with invariants and proof rules is listed below. We use `function` to define real functions ($stopDist(v)$, $accCompensation(v)$ and $SB(v)$) and boolean functions ($safeInv(m, z, v)$). The specification expresses that starting from the initial condition (pre-condition), after repeatedly running the controller (`if`, `else if`, `else` commands) and then driving according to the differential equation any number of times, the system is safe ($z \leq m$ in post-condition).

To prove safety, the loop is annotated with invariants ($v \geq 0$ and $safeInv(m, z, v)$), and the differential equation is annotated with invariant ($m - z \geq v^2/(2 * b)$) and the proof rule (`sln`). The abbreviation `sln` represents using the solution of the differential equation for proof. HHLPy will then generate verification conditions for proof and call Z3 or Wolfram Engine to verify these verification conditions.

```
function stopDist(v) = v^2/(2*b);
function accCompensation(v) = ((A/b) + 1)*((A/2)*ep^2 + ep*v);
function SB(v) = stopDist(v) + accCompensation(v);

function safeInv(m, z, v) = m-z >= stopDist(v);


pre [v >= 0] [m-z >= stopDist(v)] [b>0] [A>=0] [ep>=0];
{
    # The controller
    if (m - z < SB(v)){
        a := -b;
    }
    else if (m - z > SB(v)) {
        a := A;
    }
    else {
        a := -b; ++ a := A;
    }

    # The plant
    t := 0;
    {z_dot=v, v_dot=a, t_dot=1 & v > 0 && t < ep}     # Drive forwards for ep time (v > 0 && t == ep)
        ↪ , or drive forwards until v == 0, for at most ep time (v == 0 && t <= ep)
        invariant [m-z >= v^2/(2*b)]{sln};                # Invariant with its proof rule
}* invariant [v >= 0] [safeInv(m, z, v)];
post [z <= m];
```

IsaVODEs Our formalisation of the European Train Control System (ETCS) resembles that of KeYmaera X. We begin each problem by describing its set of variables and constants.

dataspace *ETCS* =
constants
ε :: *real* — control cycle duration upper bound
b :: *real* — braking force

A :: *real* — maximum acceleration
m :: *real* — end of movement authority (train must not drive past m)
variables
t :: *real* — Actual control cycle duration t <= ep)
z :: *real* — Train position)
v :: *real* — Current velocity)
a :: *real* — Actual acceleration -b <= a <= A)

Next, we formalise the definitions with the command **abbreviation** as shown below.

abbreviation *stopDist w* ≡ $w^2/(2*b)$

Once all definitions are in place, we provide the solution to the problem's system of ODEs, and show that it is indeed the unique solution to it.

lemma *local-flow-LICS1*: *local-flow-on* [$t \rightsquigarrow 1$, $v \rightsquigarrow \$a$, $z \rightsquigarrow \$v$] ($z +_L v +_L t$) *UNIV UNIV*
($\lambda\tau$. [$t \rightsquigarrow \tau + t$, $z \rightsquigarrow \$a * \tau^2 / 2 + \$v * \tau + \$z$, $v \rightsquigarrow \$a * \tau + \$v$])
apply (*clarsimp simp add*: *local-flow-on-def*)
apply (*unfold-locales*; *expr-simp*)
by (*rule c1-implies-local-lipschitz*[*of UNIV UNIV* - (λ(*t*::*real*,*c*). *Blinfun* (λ*c*. (*fst* (*snd c*), *0*)))])
(*auto intro*!: *has-derivative-Blinfun derivative-eq-intros poly-derivatives*)

Finally, we use this result to prove the safety specification applying Hoare and wlp-laws.

lemma *initial* ≤ |*LOOP ctrl*;*drive INV* @*loopInv*] ($z \leq m$)
apply (*subst change-loopI*[**where** *I*=(@*loopInv* ∧ $b > 0 \wedge A \geq 0 \wedge \varepsilon \geq 0$)^e])
apply (*rule hoare-loopI*)
using *ETCS-arith1*[*of b A* get_v - - ε *m* get_z -]
by (*auto simp*: *unrest-ssubst var-alpha-combine wp usubst usubst-eval*
fbox-g-dL-easiest[*OF local-flow-LICS1*] *field-simps taut-def*)
(*smt* (*verit*, *best*) *mult-left-le-imp-le zero-le-square*)

6 Conclusion and Outlook

The hybrid systems theorem proving friendly competition focuses on the characteristic features of hybrid systems theorem proving: flexibility of programming language principles for hybrid systems, unambiguous program semantics, and mathematically rigorous logical reasoning principles.

The automation tactic simplifications, nonlinear invariant generator improvements, and concurrent arithmetic backend utilization make a difference on some examples and especially in pure continuous systems verification performance, but their potential is not yet truly realized in case study verification performance. Future competitions are planned to extend the case study sub-category with game examples [CMP23] to provide better assessment of verification performance on realistic examples, and to gain insight into potential proof automation to generalize the current specialized tactics and proof scripts from single example applicability to general-purpose proof automation. A related challenge for proof repeatability and transferability are timeouts used in proof automation to decide how long to explore specific proof alternatives, and overall proof timeouts as used in this competition.

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