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Towards Computing Approximate Game Value in One Class of Infinite-Horizon Cheap Control Linear-Quadratic Differential Games: Asymptotic Analysis Approach

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Abstract

An infinite-horizon zero-sum linear-quadratic differential game (LQDG) is considered. The case, where the control cost of the minimizing player is much smaller than the control cost of the maximizing player and the state cost, is treated. The smallness of the control cost of the minimizing player is due to the presence of a positive small multiplier (parameter) in this cost. Thus, the considered differential game is a cheap control game. The second important feature of the game is that the problem dynamics have both slow and fast state variables, and the current quadratic cost of the fast state variable in the cost functional is a positive semi-definite (but non-zero) quadratic form. By the solvability conditions, the solution of the considered cheap control linear-quadratic differential game is reduced to the solution of the game-theoretic matrix Riccati algebraic equation perturbed by the small parameter. This equation represents a new type of parameter-dependent matrix Riccati algebraic equations arising in the control/differential games theory. The game features require a significantly novel approach to asymptotic analysis of this matrix Riccati algebraic equation. The zero-order asymptotic solution to this equation is derived. Using the asymptotic solution to the matrix Riccati algebraic equation, the parameter-free conditions, providing the existence of the solution to the considered cheap control differential game for all sufficiently small values of the parameter, are established. Based on the asymptotic solution to the matrix Riccati algebraic equation, two methods of approximation of the game value are proposed. The estimates of the errors in these approximations are derived. Examples, illustrating theoretical and computational aspects of the paper, are presented.

1 Introduction

We study a two-person infinite-horizon zero-sum linear-quadratic differential game. This game has the following features: (i) the control cost of the minimizing player (the minimizer) in the game's cost functional is small in comparison with the control cost of the maximizing player (the

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maximizer) and with the costs of the state variables; (ii) the matrix-valued coefficient of the cost of the fast state variable in the cost functional is a positive semi-definite (but non-zero) matrix. The first feature means that the considered game is a cheap control game. Cheap control problems/games have a considerable importance in such topics as: (1) singular controls and arcs (see, e.g, [7] and references therein); (2) limiting forms and maximally achievable accuracy of optimal regulators and filters (see, e.g., [2, 12] and references therein); (3) inverse optimal control problems (see e.g. [13]); (4) high gain control problems (see, e.g., [11] and references therein). Due to the smallness of the control cost, the differential/algebraic equation, associated with a cheap control problem/game by solvability conditions, is perturbed by a positive small parameter. This feature means that cheap control problems/games are sources of novel classes of differential/algebraic equations perturbed by positive small parameters.

As it is aforementioned, here we study a cheap control differential game. Cheap control differential games are extensively investigated in the literature. The review of this topic can be found in [8]. In most of the works, devoted to this topic, two types of the quadratic cost of the fast state variable in the integral part of the cost functional are considered: (a) the cost is a positive definite quadratic form; (b) the cost is zero. In this paper, we study an intermediate case, i.e., the case where the quadratic cost of the fast state variable in the integrand of the cost functional is a positive semi-definite (but non-zero) quadratic form. More precisely, in this paper, we consider the infinite-horizon zero-sum linear-quadratic differential game with the cheap control of the minimizer. The game's dynamics have both, slow and fast, state variables. The cost of the fast state variable in the integrand of the game's cost functional is a positive semi-definite (but non-zero) quadratic form. Saddle-point state-feedback solution of the game, considered in the present paper, is reduced to the solution of the game-theoretic matrix Riccati algebraic equation perturbed by a positive small parameter. Zero-order asymptotic solution of this equation is derived. The derivation of this asymptotic solution is based on the previous results of the authors (see [9]), and this derivation is a significant generalization of the method of asymptotic solution of the parameter-dependent matrix Riccati algebraic equation known in the literature (see, e.g., [7, 14, 15, 16] and references therein). Based on the aforementioned asymptotic solution to the game-theoretic Riccati algebraic equation, two methods of asymptotic approximation of the game's value are developed. Illustrative examples are presented.

The paper is organized as follows. In Section 2, the cheap control differential game is rigorously formulated. Main definitions are presented. In Section 3, some preliminary results are presented. These results include: (a) the parameter-dependent solvability conditions of the original game; (b) asymptotic analysis and solution of the parameter-dependent matrix Riccati algebraic equation arising in these solvability conditions; (c) derivation and analysis of a reduced differential game associated with the original one. In Section 4, based on the results of the previous section, two methods of the approximation of the value of the original cheap control differential game are developed. Errors of these approximations are derived. In Section 5 examples, illustrating theoretical and computational aspects of the paper, are considered. Conclusions are presented in Section 6.

The main notations, applied in the paper, are:

(I) $\mathbb{R}^n$ is the n-dimensional real Euclidean space;

(II) $\|\cdot\|$ is the Euclidean norm either of a vector, or of a matrix;

(III) $L_2[0,+\infty;\mathbb{R}^n]$ denotes the linear space of the functions $f(t):[0,+\infty)\to\mathbb{R}^n$ square integrable in the interval $[0,+\infty)$;

(IV) upper index T denotes the transposition either of a vector, or of a matrix;

(V) I_n is the identity matrix of dimension n;

(VI) $\mathrm{col}\big(x_1, x_2, ..., x_k\big)$, where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$,..., $x_k \in \mathbb{R}^{n_k}$, denotes the column block-vector of the dimension $n_1 + n_2 + ... + n_k$ with the upper block x_1, the next block x_2 and so on, and the lower block x_k.

2 Problem formulation

Consider the system

$$\dot{z} = Az + B_{\mathrm{u}}u + B_{\mathrm{v}}v, \quad t \geq 0, \quad z(0) = z_0, \tag{1}$$

where $z = z(t) \in \mathbb{R}^n$ is the state vector, $u = u(t) \in \mathbb{R}^r$ $(1 < r < n)$, $v = v(t) \in \mathbb{R}^s$ $(s \leq n)$ are the players' controls; A and B_i, $i = \mathrm{u}, \mathrm{v}$, are given constant matrices of corresponding dimensions; $z_0 \in \mathbb{R}^n$ is a given vector, the matrix B_{u} has the block form

$$B_{\mathrm{u}} = \begin{bmatrix} 0 \\ I_r \end{bmatrix}. \tag{2}$$

Assuming that $u(t) \in L_2[0, +\infty; \mathbb{R}^r]$ and $v(t) \in L_2[0, +\infty; \mathbb{R}^s]$, we consider the cost functional

$$J(u, v) = \int_0^{+\infty} \big[z^T(t)Dz(t) + \varepsilon^2 u^T(t)u(t) - v^T(t)G_{\mathrm{v}}v(t)\big]dt, \tag{3}$$

where $\varepsilon > 0$ is a small parameter, the constant matrix G_{v} is symmetric and positive definite, the constant matrix D has the block form

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \tag{4}$$

the $(n-r)\times(n-r)$-matrix D_1 and the $r \times r$-matrix D_2 are symmetric and positive semi-definite.

Remark 1. *In the problem with the equation of dynamics (1) and the cost functional (3) the aim of the player with the control $u(\cdot)$, called the minimizer, is to minimize the functional, while the aim of the player with the control $v(\cdot)$, called the maximizer, is to maximize the functional. We assume that both players know perfectly the data of the problem (1), (3), as well as the current state $z = z(t)$ of the system (1). Thus, the problem, consisting of the equation of dynamics (1), the cost functional (3), the above stated aims of the players and the information pattern, is an infinite-horizon zero-sum linear-quadratic differential game. Due to the smallness of the parameter $\varepsilon > 0$, the differential game (1), (3) is a minimizer's cheap control differential game, i.e., the game where the control cost of the minimizing player in the cost functional is much smaller than the control cost of the maximizing player and the state cost. In what follows, we call the game (1), (3) as the Cheap Control Infinite-Horizon Game (CCIHG).*

To define the state-feedback solution of the game (1),(3), first, we consider two sets of functions U and V. Namely, U is the set of all functions $u = u(z) : \mathbb{R}^n \to \mathbb{R}^r$ satisfying the local Lipschitz condition, while V is the set of all functions $v = v(z) : \mathbb{R}^n \to \mathbb{R}^s$ satisfying the local Lipschitz condition.

Based on the results of [7], we make the following definitions.

Definition 1. *By $\big(UV\big)$, let us denote the set of all pairs $\big(u(z),v(z)\big)$, such that the following conditions are valid: (a) $u(z)\in U$, $v(z)\in V$; (b) the initial-value problem (1) for $u=u(z)$, $v=v(z)$ and any $z_0\in\mathbb{R}^n$ has the unique solution $z_{uv}(t;z_0)$ in the entire interval $[0,+\infty)$; (c) $z_{uv}(t;z_0)\in L_2[0,+\infty;\mathbb{R}^n]$; (d) $u\big(z_{uv}(t;z_0)\big)\in L_2[0,+\infty;\mathbb{R}^r]$; (e) $v\big(z_{uv}(t;z_0)\big)\in L_2[0,+\infty;\mathbb{R}^s]$. Thus, $\big(UV\big)$ is called the set of all admissible pairs of the players' state-feedback controls in the game (1), (3).*

Now, for a given $u(z)\in U$, let us consider the set $F_{\rm v}\big(u(z)\big)\stackrel{\triangle}{=}\big\{v(z)\in V:\big(u(z),v(z)\big)\in\big(UV\big)\big\}$, and let $H_{\rm u}\stackrel{\triangle}{=}\big\{u(z)\in U:F_{\rm v}\big(u(z)\big)\neq\emptyset\big\}$.

Similarly, for a given $v(z)\in V$, let us consider the set $F_{\rm u}\big(v(z)\big)\stackrel{\triangle}{=}\big\{u(z)\in U:\big(u(z),v(z)\big)\in\big(UV\big)\big\}$, and let $H_{\rm v}\stackrel{\triangle}{=}\big\{v(z)\in V:F_{\rm u}\big(v(z)\big)\neq\emptyset\big\}$.

Definition 2. *For a given control $u(z)\in H_{\rm u}$, the value*

$$J_{\rm u}\big(u(z);z_0\big)=\sup_{v(z)\in F_{\rm v}\left(u(z)\right)}J\big(u(z),v(z)\big)$$

is called the guaranteed result of $u(z)$ in the game (1),(3).

Definition 3. *For a given control $v(z)\in H_{\rm v}$, the value*

$$J_{\rm v}\big(v(z);z_0\big)=\inf_{u(z)\in F_{\rm u}\left(v(z)\right)}J\big(u(z),v(z)\big)$$

is called the guaranteed result of $v(z)$ in the game (1),(3).

Definition 4. *We call the pair $\big(u^*(z),v^*(z)\big)\in\big(UV\big)$ a saddle-point equilibrium (or, briefly, a saddle-point) of the game (1), (3) if, for any $z_0\in\mathbb{R}^n$, the following equality is valid:*

$$J_{\rm u}\big(u^*(z);z_0\big)=J_{\rm v}\big(v^*(z);z_0\big).$$

In this case, the value

$$J^*(z_0)=J_{\rm u}\big(u^*(z);z_0\big)=J_{\rm v}\big(v^*(z);z_0\big)$$

is called a value of the game (1), (3). The solution of the initial-value problem (1) for $u=u^*(z)$, $v=v^*(z)$ is called the saddle-point trajectory of this game.

Let us represent the vectors z, z_0 and the matrices A, $B_{\rm v}$ in the block forms as:

$$z=\begin{bmatrix}x\\y\end{bmatrix},\quad z_0=\begin{bmatrix}x_0\\y_0\end{bmatrix},\quad x,x_0\in\mathbb{R}^{n-r},\quad y,y_0\in\mathbb{R}^r,$$

and

$$A=\begin{bmatrix}A_1&A_2\\A_3&A_4\end{bmatrix},\quad B_{\rm v}=\begin{bmatrix}B_{\rm v1}\\B_{\rm v2}\end{bmatrix},\tag{5}$$

where the matrices A_1, A_2, A_3 and A_4 are of the dimensions $(n-r)\times(n-r)$, $(n-r)\times r$, $r\times(n-r)$ and $r\times r$, respectively; the matrices $B_{\rm v1}$ and $B_{\rm v2}$ are of the dimensions $(n-r)\times s$ and $r\times s$, respectively.

Using these representations, as well as the equations (2) and (4), the system (1) and the cost functional (3) can be rewritten as

$$\dot{x} = A_1 x + A_2 y + B_{\mathrm{v}1} v, \quad t \geq 0, \quad x(0) = x_0, \tag{6}$$

$$\dot{y} = A_3 x + A_4 y + u + B_{\mathrm{v}2} v, \quad t \geq 0, \quad y(0) = y_0, \tag{7}$$

$$J(u,v) = \int_0^{+\infty} \left[x^T(t) D_1 x(t) + y^T(t) D_2 y(t) + \varepsilon^2 u^T(t) u(t) - v^T(t) G_{\mathrm{v}} v(t)\right] dt. \tag{8}$$

Remark 2. *Using the non-singular control transformation $\widetilde{u}(t) = \varepsilon u(t)$, ($\widetilde{u}(t)$ is a new control), the differential game (6)–(8) is converted into the equivalent differential game consisting of the differential equations (6) and*

$$\varepsilon \dot{y} = \varepsilon\left[A_3 x + A_4 y + B_{\mathrm{v}2} v\right] + \widetilde{u}, \quad t \geq 0, \quad y(0) = y_0, \tag{9}$$

and of the cost functional

$$\widetilde{J}(\widetilde{u},v) = \int_0^{+\infty} \left[x^T(t) D_1 x(t) + y^T(t) D_2 y(t) + \widetilde{u}^T(t)\widetilde{u}(t) - v^T(t) G_{\mathrm{v}} v(t)\right] dt. \tag{10}$$

The dynamic system of the game (6), (9), and (10) is singularly perturbed. In this system $x(t)$ is a slow state variable and $y(t)$ is a fast state variable (see, e.g., [17]). Therefore, we call the state variables $x(t)$ and $y(t)$ of the differential game (6)–(8) a slow state variable and a fast state variable, respectively.

Remark 3. *Since the matrix D_2 is symmetric and positive semi-definite. then by virtue of the results of [1], there exists an orthogonal matrix M of the dimension $r \times r$ such that $M^T D_2 M = \Lambda \stackrel{\triangle}{=} \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_r)$, $\lambda_k \geq 0$, $k = 1, ..., r$, where λ_1, λ_2,..., λ_r are eigenvalues of D_2. Using the invertible transformations of the y-state and u-control variables $y(t) = M\bar{y}(t)$ and $u(t) = M\bar{u}(t)$ ($\bar{y}(t)$ and $\bar{u}(t)$ are new state and control variables), we convert equivalently the game (6)–(8), to the new game with the linear dynamics*

$$\begin{aligned} \dot{x} &= A_1 x + A_2 M\bar{y} + B_{\mathrm{v}1} v, \quad t \geq 0, \quad x(0) = x_0, \\ \dot{\bar{y}} &= M^T A_3 x + M^T A_4 M\bar{y} + \bar{u}(t) + M^T B_{\mathrm{v}2} v(t), \quad t \geq 0, \quad \bar{y}(0) = M^T y_0, \end{aligned} \tag{11}$$

and the quadratic functional

$$J(\bar{u},v) = \int_0^{+\infty} \left[x^T(t) D_1 x(t) + \bar{y}^T(t)\Lambda\bar{y}(t) + \varepsilon^2 \bar{u}^T(t)\bar{u}(t) - v^T(t) G_{\mathrm{v}} v(t)\right] dt. \tag{12}$$

Thus, the form of the game (11)–(12) is similar to the form of the game (6)–(8). However, there is one important difference in this similarity. Namely, the matrix-valued coefficient in the quadratic form $y^T(t)D_2 y(t)$ is any symmetric positive semi-definite matrix, while the matrix-valued coefficient in the quadratic form $\bar{y}^T(t)\Lambda\bar{y}(t)$ is a diagonal matrix with non-negative values on its main diagonal. The latter quadratic form allows to simplify the asymptotic analysis of the cheap control game. Therefore, to avoid the above presented rather complicated calculations and the additional notations (like $\bar{y}$ and $\bar{u}$), we assume that the matrix D_2 already has the diagonal form, i.e.,

$$D_2 = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_r), \quad \lambda_k \geq 0, \quad k = 1, ..., r. \tag{13}$$

Moreover, in what follows of the paper, we study the game (6)–(8) in the case where

$$\lambda_p > 0, \quad p = 1, ..., r_1 < r; \quad \lambda_q = 0, \quad q = r_1 + 1, ..., r. \tag{14}$$

2.1 Objectives of the paper

Objectives of this paper are the following:

(I) to derive an asymptotic solution to the ε-dependent game-theoretic matrix Riccati algebraic equation, arising in the solvability conditions of the CCIHG;

(II) to establish ε-independent sufficient conditions for the existence of the CCIHG value, valid for all sufficiently small $\varepsilon > 0$;

(III) to derive an approximation of the CCIHG value.

3 Preliminary results

3.1 Parameter-dependent solvability conditions of the CCIHG

We introduce the following ε-dependent Riccati algebraic equation with respect to the $n \times n$-matrix K:

$$KA + A^T K - K\big(S_{\rm u}(\varepsilon) - S_{\rm v}\big)K + D = 0, \tag{15}$$

where

$$S_{\rm u}(\varepsilon) \stackrel{\triangle}{=} \frac{1}{\varepsilon^2} B_{\rm u} B_{\rm u}^T = \frac{1}{\varepsilon^2}\begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}, \quad S_{\rm v} \stackrel{\triangle}{=} B_{\rm v} G_{\rm v}^{-1} B_{\rm v}^T = \begin{bmatrix} S_{\rm v1} & S_{\rm v2} \\ S_{\rm v2}^T & S_{\rm v3} \end{bmatrix},$$
$$S_{\rm v1} \stackrel{\triangle}{=} B_{\rm v1} G_{\rm v}^{-1} B_{\rm v1}^T, \quad S_{\rm v2} \stackrel{\triangle}{=} B_{\rm v1} G_{\rm v}^{-1} B_{\rm v2}^T, \quad S_{\rm v3} \stackrel{\triangle}{=} B_{\rm v2} G_{\rm v}^{-1} B_{\rm v2}^T.$$

We make the assumption:

(A1) The equation (15) has a real symmetric solution $K = K^*(\varepsilon)$ such that the trivial solution of each of the following systems:

$$\dot{z} = \Big(A - \big(S_{\rm u}(\varepsilon) - S_{\rm v}\big)K^*(\varepsilon)\Big)z, \quad t \geq 0, \tag{16}$$

$$\dot{z} = \big(A - S_{\rm u}(\varepsilon)K^*(\varepsilon)\big)z, \quad t \geq 0 \tag{17}$$

is asymptotically stable.

Based on the aforementioned matrix $K^*(\varepsilon)$, we design the following controls of the minimizer and the maximizer:

$$u = u_\varepsilon^*(z) = -\frac{1}{\varepsilon^2} B_{\rm u}^T K^*(\varepsilon) z \in U, \quad v = v_\varepsilon^*(z) = G_{\rm v}^{-1} B_{\rm v}^T K^*(\varepsilon) z \in V. \tag{18}$$

By virtue of the results of [7] (see Theorem 3.2), we directly have the following assertion.

Proposition 1. *Let the assumption (A1) be valid. Then*

(I) the pair of controls $\big(u_\varepsilon^*(z), v_\varepsilon^*(z)\big)$ is the saddle-point of the CCIHG;

(II) the value of this game has the form

$$J_\varepsilon^*(z_0) = J\big(u_\varepsilon^*(z), v_\varepsilon^*(z)\big) = z_0^T K^*(\varepsilon) z_0; \tag{19}$$

(III) for any $u(z) \in F_{\rm u}\big(v_\varepsilon^*(z)\big)$ and any $v(z) \in F_{\rm v}\big(u_\varepsilon^*(z)\big)$, the following saddle-point inequality is valid in the CCIHG:

$$J\big(u_\varepsilon^*(z), v(z)\big) \leq J\big(u_\varepsilon^*(z), v_\varepsilon^*(z)\big) \leq J\big(u(z), v_\varepsilon^*(z)\big);$$

(IV) $K^*(\varepsilon)$ is a positive semi-definite matrix, meaning that $J^*(z_0) \geq 0$.

3.2 Asymptotic analysis and solution of the equation (15)

Recall that we study the CCIHG subject to the form (13) of the matrix D_2 and the assumption (14) on its eigenvalues. Therefore, the asymptotic analysis of the equation (15) also is carried out subject to (13)–(14).

3.2.1 Transformation of the equation (15)

Due to the form of the matrix $S_{\rm u}(\varepsilon)$, it has the singularity for $\varepsilon = 0$. Therefore, the left-hand side of the equation (15), being dependent on $S_{\rm u}(\varepsilon)$, also has the singularity for $\varepsilon = 0$. To remove this singularity, we will transform this equation properly. Using the equations (13), (14), and (16), we look for the solution of (15) in the following block form:

$$K = K(\varepsilon) = \begin{bmatrix} K_1(\varepsilon) & \varepsilon K_2(\varepsilon) & \varepsilon K_3(\varepsilon) \\ \varepsilon K_2^T(\varepsilon) & \varepsilon K_4(\varepsilon) & \varepsilon^2 K_5(\varepsilon) \\ \varepsilon K_3^T(\varepsilon) & \varepsilon^2 K_5^T(\varepsilon) & \varepsilon^2 K_6(\varepsilon) \end{bmatrix}, \tag{20}$$

where the symmetric matrices $K_1(\varepsilon)$, $K_4(\varepsilon)$ and $K_6(\varepsilon)$ are of the dimensions $(n-r)\times(n-r)$, $r_1 \times r_1$ and $(r-r_1)\times(r-r_1)$, respectively.

Remark 4. *It should be noted that the form (20) of the solution to the ε-dependent matrix Riccati algebraic equation was proposed for the first time in [9] where a cheap control H_∞ problem with a positive semi-definite (but nonzero) quadratic cost of the fast state variable was studied.*

Following [9], we represent the matrices A and $S_{\rm v}$ in the block forms as:

$$A = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \\ \bar{A}_4 & \bar{A}_5 & \bar{A}_6 \\ \bar{A}_7 & \bar{A}_8 & \bar{A}_9 \end{bmatrix}, \tag{21}$$

$$S_{\rm v} = \begin{bmatrix} \bar{S}_{\rm v1} & \bar{S}_{\rm v2} & \bar{S}_{\rm v3} \\ \bar{S}_{\rm v2}^T & \bar{S}_{\rm v4} & \bar{S}_{\rm v5} \\ \bar{S}_{\rm v3}^T & \bar{S}_{\rm v5}^T & \bar{S}_{\rm v6} \end{bmatrix}, \tag{22}$$

where, due to (5) and (16),

$$\bar{A}_1 = A_1, \quad [\bar{A}_2, \bar{A}_3] = A_2, \quad \begin{bmatrix} \bar{A}_4 \\ \bar{A}_7 \end{bmatrix} = A_3, \quad \begin{bmatrix} \bar{A}_5 & \bar{A}_6 \\ \bar{A}_8 & \bar{A}_9 \end{bmatrix} = A_4;$$

$$\bar{S}_{\rm v1} = S_{\rm v1}, \quad [\bar{S}_{\rm v2}, \bar{S}_{\rm v3}] = S_{\rm v2}, \quad \begin{bmatrix} \bar{S}_{\rm v4} & \bar{S}_{\rm v5} \\ \bar{S}_{\rm v5}^T & \bar{S}_{\rm v6} \end{bmatrix} = S_{\rm v3}, \tag{23}$$

the blocks $\bar{A}_1$, $\bar{A}_5$, $\bar{A}_9$ and $\bar{S}_{\rm v1}$, $\bar{S}_{\rm v4}$, $\bar{S}_{\rm v6}$ are of the same dimensions as the corresponding blocks in (20).

Based on the results of [9], we derive the asymptotic solution of the equation (15) subject to the following assumption:

(A2) $\bar{A}_3 = 0$.

Remark 5. *The assumption (A2) is of a two-fold important meaning. Firstly, this assumption has a pure mathematical meaning, because its fulfillment allows us to derive the asymptotic solution of the equation (15) based on the novel block-form representation (20) of the matrix*

$K(\varepsilon)$. Namely, if the assumption (A2) is not fulfilled, the equation (15) requires another approach to its asymptotic analysis, including another (than (20)) representation of its solution. The corresponding example is presented in Section 5.2. Secondly, the assumption (A2) has a differential game-theoretic meaning which is presented in Section 3.3.

Now, using the equations (13), (14), (16), (20), (21), and (22), as well as the assumption (A2), we can transform the equation (15) (quite similarly to the results of [9]) into the following equivalent set of the equations:

$$\begin{aligned} &K_1(\varepsilon)\bar{A}_1 + \bar{A}_1^T K_1(\varepsilon) - K_2(\varepsilon)K_2^T(\varepsilon) - K_3(\varepsilon)K_3^T(\varepsilon) + K_1(\varepsilon)\bar{S}_{\mathrm{v}1}K_1(\varepsilon) \\ &\qquad + D_1 + \varepsilon P_1\big(K_1(\varepsilon), K_2(\varepsilon), K_3(\varepsilon), \varepsilon\big) = 0, \end{aligned} \tag{24}$$

$$K_1(\varepsilon)\bar{A}_2 - K_2(\varepsilon)K_4(\varepsilon) + \varepsilon P_2\big(K_1(\varepsilon), K_2(\varepsilon), K_3(\varepsilon), K_4(\varepsilon), K_5(\varepsilon), \varepsilon\big) = 0, \tag{25}$$

$$\begin{aligned} &K_2(\varepsilon)\bar{A}_6 + K_3(\varepsilon)\bar{A}_9 + \bar{A}_1^T K_3(\varepsilon) - K_2(\varepsilon)K_5(\varepsilon) - K_3(\varepsilon)K_6(\varepsilon) \\ &+ K_1(\varepsilon)\bar{S}_{\mathrm{v}1}K_3(\varepsilon) + \varepsilon P_3\big(K_1(\varepsilon), K_2(\varepsilon), K_3(\varepsilon), K_5(\varepsilon), K_6(\varepsilon), \varepsilon\big) = 0, \end{aligned} \tag{26}$$

$$-\big(K_4(\varepsilon)\big)^2 + \Lambda_1 + \varepsilon P_4\big(K_2(\varepsilon), K_4(\varepsilon), K_5(\varepsilon), \varepsilon\big) = 0, \tag{27}$$

$$K_4(\varepsilon)\bar{A}_6 + \bar{A}_2^T K_3(\varepsilon) - K_4(\varepsilon)K_5(\varepsilon) + \varepsilon P_5\big(K_2(\varepsilon), K_3(\varepsilon), K_4(\varepsilon), K_5(\varepsilon), K_6(\varepsilon), \varepsilon\big) = 0, \tag{28}$$

$$\begin{aligned} &K_5^T(\varepsilon)\bar{A}_6 + K_6(\varepsilon)\bar{A}_9 + \bar{A}_6^T K_5(\varepsilon) + \bar{A}_9^T K_6(\varepsilon) - K_5^T(\varepsilon)K_5(\varepsilon) - K_6(\varepsilon)K_6(\varepsilon) \\ &\qquad + K_3^T(\varepsilon)\bar{S}_{\mathrm{v}1}K_3(\varepsilon) + \varepsilon P_6\big(K_3(\varepsilon), K_5(\varepsilon), K_6(\varepsilon), \varepsilon\big) = 0, \end{aligned} \tag{29}$$

where

$$\begin{aligned} P_1\big(K_1(\varepsilon), K_2(\varepsilon), K_3(\varepsilon), \varepsilon\big) &\triangleq K_2(\varepsilon)\bar{A}_4 + K_3(\varepsilon)\bar{A}_7 + \bar{A}_4^T K_2^T(\varepsilon) + \bar{A}_7^T K_3^T(\varepsilon) \\ &+ K_2(\varepsilon)\bar{S}_{\mathrm{v}2}^T K_1(\varepsilon) + K_3(\varepsilon)\bar{S}_{\mathrm{v}3}^T K_1(\varepsilon) + K_1(\varepsilon)\bar{S}_{\mathrm{v}2}K_2^T(\varepsilon) + \varepsilon K_2(\varepsilon)\bar{S}_{\mathrm{v}4}K_2^T(\varepsilon) \\ &+ \varepsilon K_3(\varepsilon)\bar{S}_{\mathrm{v}5}^T K_2^T(\varepsilon) + K_1(\varepsilon)\bar{S}_{\mathrm{v}3}K_3^T(\varepsilon) + \varepsilon K_2(\varepsilon)\bar{S}_{\mathrm{v}5}k_3^T(\varepsilon) + \varepsilon K_3(\varepsilon)\bar{S}_{\mathrm{v}6}K_3^T(\varepsilon), \end{aligned}$$

$$\begin{aligned} P_2\big(K_1(\varepsilon), K_2(\varepsilon), K_3(\varepsilon), K_4(\varepsilon), K_5(\varepsilon), \varepsilon\big) &\triangleq K_2(\varepsilon)\bar{A}_5 + K_3(\varepsilon)\bar{A}_8 + \bar{A}_1^T K_2(\varepsilon) + \bar{A}_4^T K_4^T(\varepsilon) \\ &+ \varepsilon\bar{A}_7^T K_5^T(\varepsilon) - K_3(\varepsilon)K_5^T(\varepsilon) + K_1(\varepsilon)\bar{S}_{\mathrm{v}1}K_2(\varepsilon) + \varepsilon K_2(\varepsilon)\bar{S}_{\mathrm{v}2}^T K_2(\varepsilon) + \varepsilon K_3(\varepsilon)\bar{S}_{\mathrm{v}3}^T K_2(\varepsilon) \\ &+ K_1(\varepsilon)\bar{S}_{\mathrm{v}2}K_4(\varepsilon) + \varepsilon K_2(\varepsilon)\bar{S}_{\mathrm{v}4}K_4(\varepsilon) + \varepsilon K_3(\varepsilon)\bar{S}_{\mathrm{v}5}^T K_4(\varepsilon) + \varepsilon K_1(\varepsilon)\bar{S}_{\mathrm{v}3}K_5^T(\varepsilon) \\ &+ \varepsilon^2 K_2(\varepsilon)\bar{S}_{\mathrm{v}5}K_5^T(\varepsilon) + \varepsilon^2 K_3(\varepsilon)\bar{S}_{\mathrm{v}6}K_5^T(\varepsilon), \end{aligned}$$

$$\begin{aligned} P_3\big(K_1(\varepsilon), K_2(\varepsilon), K_3(\varepsilon), K_5(\varepsilon), K_6(\varepsilon), \varepsilon\big) &\triangleq \bar{A}_4^T K_5(\varepsilon) + \bar{A}_7^T K_6(\varepsilon) + K_2(\varepsilon)\bar{S}_{\mathrm{v}2}^T K_3(\varepsilon) \\ &+ K_3(\varepsilon)\bar{S}_{\mathrm{v}3}^T K_3(\varepsilon) + K_1(\varepsilon)\bar{S}_{\mathrm{v}2}K_5(\varepsilon) + \varepsilon K_2(\varepsilon)\bar{S}_{\mathrm{v}4}K_5(\varepsilon) + \varepsilon K_3(\varepsilon)\bar{S}_{\mathrm{v}5}^T K_5(\varepsilon) \\ &+ K_1(\varepsilon)\bar{S}_{\mathrm{v}3}K_6(\varepsilon) + \varepsilon K_2(\varepsilon)\bar{S}_{\mathrm{v}5}K_6(\varepsilon) + \varepsilon K_3(\varepsilon)\bar{S}_{\mathrm{v}6}K_6(\varepsilon), \end{aligned}$$

$$\begin{aligned} P_4\big(K_2(\varepsilon), K_4(\varepsilon), K_5(\varepsilon), \varepsilon\big) &\triangleq K_2^T(\varepsilon)\bar{A}_2 + K_4(\varepsilon)\bar{A}_5 + \varepsilon K_5(\varepsilon)\bar{A}_8 + \bar{A}_2^T K_2(\varepsilon) + \bar{A}_5^T K_4(\varepsilon) \\ &+ \varepsilon\bar{A}_8^T K_5^T(\varepsilon) - \varepsilon K_5(\varepsilon)K_5^T(\varepsilon) + \varepsilon K_2^T(\varepsilon)\bar{S}_{\mathrm{v}1}K_2(\varepsilon) + \varepsilon K_4(\varepsilon)\bar{S}_{\mathrm{v}2}^T K_2(\varepsilon) + \varepsilon^2 K_5(\varepsilon)\bar{S}_{\mathrm{v}3}^T K_2(\varepsilon) \\ &+ \varepsilon K_2^T(\varepsilon)\bar{S}_{\mathrm{v}2}K_4(\varepsilon) + \varepsilon K_4(\varepsilon)\bar{S}_{\mathrm{v}4}K_4(\varepsilon) + \varepsilon^2 K_5(\varepsilon)\bar{S}_{\mathrm{v}5}^T K_4(\varepsilon) + \varepsilon^2 K_2^T(\varepsilon)\bar{S}_{\mathrm{v}3}K_5^T(\varepsilon) \\ &+ \varepsilon^2 K_4(\varepsilon)\bar{S}_{\mathrm{v}5}K_5^T(\varepsilon) + \varepsilon^3 K_5(\varepsilon)\bar{S}_{\mathrm{v}6}K_5^T(\varepsilon), \end{aligned}$$

$$P_5\big(K_2(\varepsilon),K_3(\varepsilon),K_4(\varepsilon),K_5(\varepsilon),K_6(\varepsilon),\varepsilon\big)=K_5(\varepsilon)\bar{A}_9+\bar{A}_5^TK_5(\varepsilon)+\bar{A}_8^TK_6(\varepsilon)$$
$$-K_5(\varepsilon)K_6(\varepsilon)+K_2^T(\varepsilon)\bar{S}_{\mathrm{v}1}K_3(\varepsilon)+K_4(\varepsilon)\bar{S}_{\mathrm{v}2}^TK_3(\varepsilon)+\varepsilon K_5(\varepsilon)\bar{S}_{\mathrm{v}3}^TK_3(\varepsilon)$$
$$+\varepsilon K_2^T(\varepsilon)\bar{S}_{\mathrm{v}2}K_5(\varepsilon)+\varepsilon K_4(\varepsilon)\bar{S}_{\mathrm{v}4}K_5(\varepsilon)+\varepsilon^2K_5(\varepsilon)\bar{S}_{\mathrm{v}5}^TK_5(\varepsilon)$$
$$+\varepsilon K_2^T(\varepsilon)\bar{S}_{\mathrm{v}3}K_6(\varepsilon)+\varepsilon K_4(\varepsilon)\bar{S}_{\mathrm{v}5}K_6(\varepsilon)+\varepsilon^2K_5(\varepsilon)\bar{S}_{\mathrm{v}6}K_6(\varepsilon),$$

$$P_6\big(K_3(\varepsilon),K_5(\varepsilon),K_6(\varepsilon),\varepsilon\big)\stackrel{\triangle}{=}K_5^T(\varepsilon)\bar{S}_{\mathrm{v}2}^TK_3(\varepsilon)+K_6(\varepsilon)\bar{S}_{\mathrm{v}3}^TK_3(\varepsilon)+K_3^T(\varepsilon)\bar{S}_{\mathrm{v}2}K_5(\varepsilon)$$
$$+\varepsilon K_5^T(\varepsilon)\bar{S}_{\mathrm{v}4}K_5(\varepsilon)+\varepsilon K_6(\varepsilon)\bar{S}_{\mathrm{v}5}^TK_5(\varepsilon)+K_3^T(\varepsilon)\bar{S}_{\mathrm{v}3}K_6(\varepsilon)+\varepsilon K_5^T(\varepsilon)\bar{S}_{\mathrm{v}5}K_6(\varepsilon)$$
$$+\varepsilon K_6(\varepsilon)\bar{S}_{\mathrm{v}6}K_6(\varepsilon);$$

$$\Lambda_1\stackrel{\triangle}{=}\mathrm{diag}\big(\lambda_1,...,\lambda_{r_1}\big).$$

Remark 6. *The nine-block-form representation (20) of the matrix $K(\varepsilon)$ and the system (24)–(29), obtained due to this block form, differ considerably from the results of [3, 7, 14, 15, 16] where the four-block-form representation of the solution to the corresponding parameter-dependent matrix Riccati algebraic equation is used. The representation (20) and the system (24)–(29) are essentially novel results in the asymptotic analysis of parameter-dependent matrix Riccati algebraic equations with singularities, arising in infinite horizon differential games, optimal control problems and H_∞ control problems. In Section 5.3, an example is presented showing that the four-block-form representation of the matrix $K(\varepsilon)$, used in [3, 7, 14, 15, 16], is not applicable for the derivation of the asymptotic solution to the matrix Riccati algebraic equation (15) in the case of (13)–(14).*

3.2.2 Zero-order asymptotic solution of the set of the equations (24)–(29)

We look for the zero-order asymptotic solution $\{K_{10},K_{20},K_{30},K_{40},K_{50},K_{60}\}$ of the set of the equations (24)–(29). The equations for K_{l0}, $(l=1,2,...,6)$ are obtained setting formally $\varepsilon=0$ in (24)–(29) and replacing $K_l(\varepsilon)$ with K_{l0}, $(l=1,2,...,6)$. Thus, we have

$$K_{10}\bar{A}_1+\bar{A}_1^TK_{10}-K_{20}K_{20}^T-K_{30}K_{30}^T+K_{10}\bar{S}_{\mathrm{v}1}K_{10}+D_1=0, \tag{30}$$

$$K_{10}\bar{A}_2-K_{20}K_{40}=0, \tag{31}$$

$$K_{20}\bar{A}_6+K_{30}\bar{A}_9+\bar{A}_1^TK_{30}-K_{20}K_{50}-K_{30}K_{60}+K_{10}\bar{S}_{\mathrm{v}1}K_{30}=0, \tag{32}$$

$$-\big(K_{40}\big)^2+\Lambda_1=0, \tag{33}$$

$$K_{40}\bar{A}_6+\bar{A}_2^TK_{30}-K_{40}K_{50}=0, \tag{34}$$

$$K_{50}^T\bar{A}_6+K_{60}\bar{A}_9+\bar{A}_6^TK_{50}+\bar{A}_9^TK_{60}-K_{50}^TK_{50}-\big(K_{60}\big)^2+K_{30}^T\bar{S}_{\mathrm{v}1}K_{30}=0. \tag{35}$$

Let us introduce the matrices

$$\Lambda_1^{1/2}=\mathrm{diag}(\lambda_1^{1/2},...,\lambda_{r_1}^{1/2}),\qquad \Lambda_1^{-1/2}=\mathrm{diag}(\lambda_1^{-1/2},...,\lambda_{r_1}^{-1/2}).$$

Due to the results of [9], we have the following components of the solution of the set (30)–(35):

$$K_{20} = K_{10}\bar{A}_2\Lambda_1^{-1/2}, \tag{36}$$

$$K_{30} = 0, \tag{37}$$

$$K_{40} = \Lambda_1^{1/2}, \tag{38}$$

$$K_{50} = \bar{A}_6. \tag{39}$$

Furthermore, due to [9], the components K_{10} and K_{60} of the solution to the set (30)–(35) satisfy the following matrix algebraic Riccati equations:

$$K_{10}\bar{A}_1 + \bar{A}_1^T K_{10} - K_{10}\left(\bar{A}_2\Lambda_1^{-1}\bar{A}_2^T - \bar{S}_{\mathrm{v}1}\right)K_{10} + D_1 = 0, \tag{40}$$

$$K_{60}\bar{A}_9 + \bar{A}_9{}^T K_{60} - K_{60}^2 + \bar{A}_6^T\bar{A}_6 = 0. \tag{41}$$

We make the following assumptions with respect to these equations:

(**A3**) The equation (40) has a real symmetric solution K_{10} such that the trivial solution of each of the following systems:

$$\dot{\bar{x}} = \left(\bar{A}_1 - \left(\bar{A}_2\Lambda_1^{-1}\bar{A}_2^T - \bar{S}_{\mathrm{v}1}\right)K_{10}\right)\bar{x}, \quad t \geq 0,$$

$$\dot{\bar{x}} = \left(\bar{A}_1 - \bar{A}_2\Lambda_1^{-1}\bar{A}_2^T K_{10}\right)\bar{x}, \quad t \geq 0$$

is asymptotically stable.

(**A4**)

$$\operatorname{rank}\left[\bar{A}_6^T, \bar{A}_9^T\bar{A}_6^T, ..., \left(\bar{A}_9^T\right)^{r-r_1-1}\bar{A}_6^T\right] = r - r_1. \tag{42}$$

Lemma 1. *Let the assumption (A4) be valid. Then, there exists the symmetric positive definite solution K_{60} to the equation (41), and the trivial solution of the system*

$$\dot{\bar{y}} = \left(\bar{A}_9 - K_{60}\right)\bar{y}, \quad t \geq 0$$

is asymptotically stable.

Proof. Let us consider the optimal control problem, consisting of the dynamic system

$$\dot{\bar{y}} = \bar{A}_9\bar{y} + \bar{u}, \quad t \geq 0, \quad \bar{y}(0) = \bar{y}_0, \tag{43}$$

and the performance index

$$\bar{J}(\bar{u}) = \int_0^{+\infty}\left[\bar{y}^T(t)\bar{A}_6^T\bar{A}_6\bar{y}(t) + \bar{u}^T(t)\bar{u}(t)\right]dt \to \min_{\bar{u}(\cdot)}, \tag{44}$$

where $\bar{y} = \bar{y}(t) \in \mathbb{R}^{r-r_1}$ is the state vector, $\bar{u} = \bar{u}(t) \in \mathbb{R}^{r-r_1}$ is the control, $\bar{y}_0 \in \mathbb{R}^{r-r_1}$ is a given constant vector.

The equality (42), along with the results of [10], directly yields the existence of the unique symmetric positive definite solution K_{60} to the equation (41). The solution (the optimal state-feedback control) of the problem (43)–(44) has the form

$$\bar{u}(\bar{y}) = -K_{60}\bar{y},$$

and this control stabilizes the system (43). This completes the proof of the lemma. □

3.2.3 Justification of the asymptotic solution to the set (24)–(29)

Lemma 2. *Let the assumptions (A2)–(A4) be valid. Then, there exists a positive number ε_0 such that, for all $\varepsilon \in (0, \varepsilon_0]$, the set of equations (24)–(29) has the real solution $K_\ell(\varepsilon) = K^*_\ell(\varepsilon)$, $(\ell = 1, ..., 6)$, satisfying the property $\left(K^*_{\ell_1}(\varepsilon)\right)^T = K^*_{\ell_1}(\varepsilon)$, $(\ell_1 = 1, 4, 6)$, and the inequalities*

$$\left\|K^*_\ell(\varepsilon) - K_{\ell 0}\right\| \leq a\varepsilon, \qquad \ell = 1, ..., 6, \tag{45}$$

where $a > 0$ is some constant independent of ε.

Proof. The lemma is proved in the lines of the proof of [9, Lemma 3]. □

Consider the block-form matrix

$$K^*(\varepsilon) \triangleq \begin{bmatrix} K^*_1(\varepsilon) & \varepsilon K^*_2(\varepsilon) & \varepsilon K^*_3(\varepsilon) \\ \varepsilon\left(K^*_2(\varepsilon)\right)^T & \varepsilon K^*_4(\varepsilon) & \varepsilon^2 K^*_5(\varepsilon) \\ \varepsilon\left(K^*_3(\varepsilon)\right)^T & \varepsilon^2\left(K^*_5(\varepsilon)\right)^T & \varepsilon^2 K^*_6(\varepsilon) \end{bmatrix}, \qquad \varepsilon \in (0, \varepsilon_0], \tag{46}$$

where $K^*_\ell(\varepsilon)$, $(\ell = 1, ..., 6)$, is the solution of the set of the equations (24)–(29) mentioned in Lemma 2.

Corollary 1. *Let the assumptions (A2)–(A4) be valid. Then, there exists a number $0 < \varepsilon^* \leq \varepsilon_0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, the matrix (46) satisfies the assumption (A1). Thus, for these ε, all the statements of Proposition 1 are valid, meaning that the CCIHG value exists and has the form (19).*

Proof. Remember that the set of the equations (24)–(29) is equivalent to the equation (15). Furthermore, $K^*_\ell(\varepsilon)$, $(\ell = 1, ..., 6)$ is a real solution of (24)–(29) for all $\varepsilon \in (0, \varepsilon_0]$. Therefore, the matrix $K^*(\varepsilon)$, given by (46), is a real symmetric solution of (15).

Using Lemma 1, the asymptotic stability of the trivial solution to the systems (16) and (17) for all sufficiently small $\varepsilon > 0$ is proved quite similarly to the results of [9, Corollary 1 and its proof]. This completes the proof of the corollary. □

3.3 Reduced differential game

In this section, we derive a new differential game associated with the original CCIHG. This new game has two features: (a) it is independent of the parameter ε; (b) the Euclidean space dimension of the dynamics in the new game is smaller than such a dimension in the CCIHG. The new game is constructed by the algorithm proposed in [5, 6] and then applied in [9].

First, we transform the CCIHG to the differential game, consisting of the singularly perturbed system (6), (9) and the non-cheap control cost functional (10) (see Remark 2). Then, setting formally $\varepsilon = 0$ in the game (6), (9), and (10) and re-denoting the notations x, y, $\widetilde{u}$, v, $\widetilde{J}$ in the resulting system and functional with $x_{\rm r}$, $y_{\rm r}$, $\widetilde{u}_{\rm r}$, $v_{\rm r}$, $\widetilde{J}_{\rm r}$, respectively, we obtain

$$\dot{x}_{\rm r} = A_1 x_{\rm r} + A_2 y_{\rm r} + B_{\rm v1} v_{\rm r}, \quad t \geq 0, \quad x_{\rm r}(0) = x_0, \tag{47}$$

$$\widetilde{u}_{\rm r}(t) = 0, \qquad t \geq 0, \tag{48}$$

$$\widetilde{J}_{\rm r} = \int_0^{+\infty} \left[x^T_{\rm r}(t) D_1 x_{\rm r}(t) + y^T_{\rm r}(t) D_2 y_{\rm r}(t) + \widetilde{u}^T_{\rm r} \widetilde{u}_{\rm r}(t) - v^T_{\rm r}(t) G_{\rm v} v_{\rm r}(t)\right] dt. \tag{49}$$

Substituting (48) into the functional (49), we obtain

$$\widetilde{J}_{\mathrm{r}} = \int_0^{+\infty} \left[x_{\mathrm{r}}^T(t) D_1 x_{\mathrm{r}}(t) + y_{\mathrm{r}}^T(t) D_2 y_{\mathrm{r}}(t) - v_{\mathrm{r}}^T(t) G_{\mathrm{v}} v_{\mathrm{r}}(t)\right] dt. \tag{50}$$

In the set (47), (50), the variable $y_{\mathrm{r}}(t)$, $t \geq 0$ does not satisfy any equation. Due to this feature of $y_{\mathrm{r}}(t)$, we can choose it to satisfy a desirable property of the system (47) and the functional (50). The latter means that $y_{\mathrm{r}}(t)$ can be chosen as a control in these system and functional. In such a case, the functional (50) depends on y_{r} and v_{r}, i.e., $\widetilde{J}_{\mathrm{r}} = \widetilde{J}_{\mathrm{r}}(y_{\mathrm{r}}, v_{\mathrm{r}})$. Since v_{r} is the control of the maximizer, then it is reasonably to choose y_{r} as the control of the minimizer. Thus, the problem (47), (50) becomes an infinite-horizon zero-sum linear-quadratic differential game. In this game, both players are aware of all the game's data and of the game's current state $x_{\mathrm{r}} = x_{\mathrm{r}}(t)$.

Remember that the matrix D_2 is not invertible (see the equations (13) and (14)). Therefore, the game (47), (50) can be directly solved neither by the Isaacs's MinMax principle nor by the Bellman-Isaacs equation method, i.e., it is singular (see, e.g., [7]). However, if the assumption (A2) is fulfilled, the game (47), (50) becomes as:

$$\dot{x}_{\mathrm{r}} = A_1 x_{\mathrm{r}} + \bar{A}_2 y_{\mathrm{r},1} + B_{\mathrm{v}1} v_{\mathrm{r}}, \quad t \geq 0, \quad x_{\mathrm{r}}(0) = x_0, \tag{51}$$

$$\widetilde{J}_{\mathrm{r}}(y_{\mathrm{r},1}, v_{\mathrm{r}}) = \int_0^{+\infty} \left[x_{\mathrm{r}}^T(t) D_1 x_{\mathrm{r}}(t) + y_{\mathrm{r},1}^T(t) \Lambda_1 y_{\mathrm{r},1}(t) - v_{\mathrm{r}}^T(t) G_{\mathrm{v}} v_{\mathrm{r}}(t)\right] dt, \tag{52}$$

where $y_{\mathrm{r},1}(t) \in \mathbb{R}^{r_1}$ is the upper block of the vector $y_{\mathrm{r}}(t)$ and it is the minimizer's control in the game (51)–(52).

The matrix Λ_1 is positive definite. Therefore, the game (51)–(52) is regular, i.e., the Isaacs's MinMax principle and the Bellman-Isaacs equation method can be directly used for its solution. We call this game the Reduced Differential Game (RDG). Thus, due to the assumption (A2), the game (47), (50) becomes the regular RDG (51)–(52).

To define the state-feedback solution of the RDG, first, we consider two sets of functions Y_{r} and V_{r}. Here, Y_{r} is the set of all functions $y_{\mathrm{r},1} = y_{\mathrm{r},1}(x_{\mathrm{r}}) : \mathbb{R}^{n-r} \to \mathbb{R}^{r_1}$ satisfying the local Lipschitz condition, while V_{r} is the set of all functions $v_{\mathrm{r}} = v_{\mathrm{r}}(x_{\mathrm{r}}) : \mathbb{R}^{n-r} \to \mathbb{R}^s$ satisfying the local Lipschitz condition.

Similarly to Definitions 1–4, we introduce the following definitions associated with the RDG.

Definition 5. *By $\big(YV\big)_{\mathrm{r}}$, let us denote the set of all pairs $\big(y_{\mathrm{r},1}(x_{\mathrm{r}}), v_{\mathrm{r}}(x_{\mathrm{r}})\big)$, such that the following conditions are valid: (a_{r}) $y_{\mathrm{r},1}(x_{\mathrm{r}}) \in Y_{\mathrm{r}}$, $v_{\mathrm{r}}(x_{\mathrm{r}}) \in V_{\mathrm{r}}$; ($\mathrm{b}_{\mathrm{r}}$) the initial-value problem (51) for $y_{\mathrm{r},1} = y_{\mathrm{r},1}(x_{\mathrm{r}})$, $v_{\mathrm{r}} = v_{\mathrm{r}}(x_{\mathrm{r}})$ and any $x_0 \in \mathbb{R}^{n-r}$ has the unique solution $x_{\mathrm{r}}(t; x_0)$ in the entire interval $[0, +\infty)$; (c_{r}) $x_{\mathrm{r}}(t; x_0) \in L_2[0, +\infty; \mathbb{R}^{n-r}]$; ($\mathrm{d}_{\mathrm{r}}$) $y_{\mathrm{r},1}\big(x_{\mathrm{r}}(t; x_0)\big) \in L_2[0, +\infty; \mathbb{R}^{r_1}]$; ($\mathrm{e}_{\mathrm{r}}$) $v_{\mathrm{r}}\big(x_{\mathrm{r}}(t; x_0)\big) \in L_2[0, +\infty; \mathbb{R}^s]$. Thus, $\big(YV\big)_{\mathrm{r}}$ is called the set of all admissible pairs of the players' state-feedback controls in the RDG.*

Now, for a given $y_{\mathrm{r},1}(x_{\mathrm{r}}) \in Y_r$, let us consider the set $F_{\mathrm{r,v}}\big(y_{\mathrm{r},1}(x_{\mathrm{r}})\big) \triangleq \big\{v_{\mathrm{r}}(x_{\mathrm{r}}) \in V_{\mathrm{r}} : \big(y_{\mathrm{r},1}(x_{\mathrm{r}}), v_{\mathrm{r}}(x_{\mathrm{r}})\big) \in \big(YV\big)_{\mathrm{r}}\big\}$, and let $H_{\mathrm{r,y}} \triangleq \big\{y_{\mathrm{r},1}(x_{\mathrm{r}}) \in Y_{\mathrm{r}} : F_{\mathrm{r,v}}\big(y_{\mathrm{r},1}(x_{\mathrm{r}})\big) \neq \emptyset\big\}$.

Similarly, for a given $v_{\mathrm{r}}(x_{\mathrm{r}}) \in V_{\mathrm{r}}$, let us consider the set $F_{\mathrm{r,y}}\big(v_{\mathrm{r}}(x_{\mathrm{r}})\big) \triangleq \big\{y_{\mathrm{r},1}(x_{\mathrm{r}}) \in Y_{\mathrm{r}} : \big(y_{\mathrm{r},1}(x_{\mathrm{r}}), v_{\mathrm{r}}(x_{\mathrm{r}})\big) \in \big(YV\big)_{\mathrm{r}}\big\}$, and let $H_{\mathrm{r,v}} \triangleq \big\{v_{\mathrm{r}}(x_{\mathrm{r}}) \in V_{\mathrm{r}} : F_{\mathrm{r,y}}\big(v_{\mathrm{r}}(x_{\mathrm{r}})\big) \neq \emptyset\big\}$.

Definition 6. *For a given control $y_{\mathrm{r},1}(x_{\mathrm{r}}) \in H_{\mathrm{r,y}}$, the value*

$$\widetilde{J}_{\mathrm{r,y}}\big(y_{\mathrm{r},1}(x_{\mathrm{r}}); x_0\big) = \sup_{v_{\mathrm{r}}(x_{\mathrm{r}}) \in F_{\mathrm{r,v}}\left(y_{\mathrm{r},1}(x_{\mathrm{r}})\right)} \widetilde{J}_{\mathrm{r}}\big(y_{\mathrm{r},1}(x_{\mathrm{r}}), v_{\mathrm{r}}(x_{\mathrm{r}})\big)$$

is called the guaranteed result of $y_{\mathrm{r},1}(x_\mathrm{r})$ in the RDG.

Definition 7. *For a given control $v_\mathrm{r}(x_\mathrm{r}) \in H_{\mathrm{r,v}}$, the value*

$$\widetilde{J}_{\mathrm{r,v}}\big(v_\mathrm{r}(x_\mathrm{r}); x_0\big) = \inf_{y_{\mathrm{r},1}(x_\mathrm{r}) \in F_{\mathrm{r,y}}\left(v_\mathrm{r}(x_\mathrm{r})\right)} \widetilde{J}_\mathrm{r}\big(y_{\mathrm{r},1}(x_\mathrm{r}), v_\mathrm{r}(x_\mathrm{r})\big)$$

is called the guaranteed result of $v_\mathrm{r}(z)$ in the RDG.

Definition 8. *We call the pair $\big(y^*_{\mathrm{r},1}(x_\mathrm{r}), v^*_\mathrm{r}(x_\mathrm{r})\big) \in \big(YV\big)_\mathrm{r}$ a saddle-point equilibrium (or, briefly, a saddle-point) of the RDG if, for any $x_0 \in \mathbb{R}^{n-r}$, the following equality is valid:*

$$\widetilde{J}_{\mathrm{r,y}}\big(y^*_{\mathrm{r},1}(x_\mathrm{r}); x_0\big) = \widetilde{J}_{\mathrm{r,v}}\big(v^*_\mathrm{r}(x_\mathrm{r}); x_0\big).$$

In this case, the value

$$\widetilde{J}^*_\mathrm{r}(x_0) = \widetilde{J}_{\mathrm{r,y}}\big(y^*_{\mathrm{r},1}(x_\mathrm{r}); x_0\big) = \widetilde{J}_{\mathrm{r,v}}\big(v^*_\mathrm{r}(x_\mathrm{r}); x_0\big)$$

is called a value of the RDG. The solution of the initial-value problem (51) for $y_{\mathrm{r},1} = y^*_{\mathrm{r},1}(x_\mathrm{r})$, $v_\mathrm{r} = v^*_\mathrm{r}(x_\mathrm{r})$ is called the saddle-point trajectory of this game.

Consider the functions

$$y_{\mathrm{r},1} = y^*_{\mathrm{r},1}(x_\mathrm{r}) = -\Lambda^{-1}\bar{A}_2 K_{10} x_\mathrm{r}, \quad v_\mathrm{r} = v^*_\mathrm{r}(x_\mathrm{r}) = G_\mathrm{v}^{-1} B_{\mathrm{v}1} K_{10} x_\mathrm{r},$$

where K_{10} is the solution of the problem (40) mentioned in the assumption (A3).

Taking into account that $\bar{S}_{\mathrm{v}1} = S_{\mathrm{v}1} = B_{\mathrm{v}1} G_\mathrm{v}^{-1} B_{\mathrm{v}1}^T$ (see the equations (16) and (23)), we have (quite similarly to Proposition 1) the following assertion.

Proposition 2. *Let the assumption (A3) be valid. Then*

(I_r) the pair of controls $\big(y^*_{\mathrm{r},1}(x_\mathrm{r}), v^*_\mathrm{r}(x_\mathrm{r})\big)$ is the saddle-point of the RDG;

(II_r) the value of this game has the form

$$\widetilde{J}^*_\mathrm{r}(x_0) = \widetilde{J}_\mathrm{r}\big(y^*_{\mathrm{r},1}(x_\mathrm{r}), v^*_\mathrm{r}(x_\mathrm{r})\big) = x_0^T K_{10} x_0; \tag{53}$$

(III_r) for any $y_{\mathrm{r},1}(x_\mathrm{r}) \in F_{\mathrm{r,y}}\big(v^*_\mathrm{r}(x_\mathrm{r})\big)$ and any $v_\mathrm{r}(x_\mathrm{r}) \in F_{\mathrm{r,v}}\big(y^*_{\mathrm{r},1}(x_\mathrm{r})\big)$, the following saddle-point inequality is valid in the RDG:

$$\widetilde{J}_\mathrm{r}\big(y^*_{\mathrm{r},1}(x_\mathrm{r}), v_\mathrm{r}(x_\mathrm{r})\big) \le \widetilde{J}_\mathrm{r}\big(y^*_{\mathrm{r},1}(x_\mathrm{r}), v^*_\mathrm{r}(x_\mathrm{r})\big) \le \widetilde{J}_\mathrm{r}\big(y_{\mathrm{r},1}(x_\mathrm{r}), v^*_\mathrm{r}(x_\mathrm{r})\big);$$

(IV_r) K_{10} is a positive semi-definite matrix, meaning that $\widetilde{J}^*_\mathrm{r}(x_0) \ge 0$.

4 Main results: Approximation of the CCIHG value

The approximation of the CCIHG value is based on the following matrix:

$$K_0(\varepsilon) = \begin{bmatrix} K_{10} & \varepsilon K_{20} & \varepsilon K_{30} \\ \varepsilon K_{20}^T & \varepsilon K_{40} & \varepsilon^2 K_{50} \\ \varepsilon K_{30}^T & \varepsilon^2 K_{50}^T & \varepsilon^2 K_{60} \end{bmatrix}, \tag{54}$$

where the matrices K_{l0}, $(l = 1, 2, ..., 6)$, are given by (36)–(41).

Lemma 3. *Let the assumptions (A2)–(A4) be valid. Then, there exists a number* $0 < \varepsilon_1^* \leq \varepsilon^*$ *such that, for all* $\varepsilon \in (0, \varepsilon_1^*]$*, the trivial solution of each of the following systems:*

$$\dot{z} = \Big(A - \big(S_{\rm u}(\varepsilon) - S_{\rm v}\big)K_0(\varepsilon)\Big)z, \quad t \geq 0, \tag{55}$$

$$\dot{z} = \big(A - S_u(\varepsilon)K_0(\varepsilon)\big)z, \quad t \geq 0 \tag{56}$$

is asymptotically stable.

Proof. Using Lemma 1, the asymptotic stability of the trivial solution to the systems (55) and (56) for all sufficiently small $\varepsilon > 0$ is proved quite similarly to the results of [9, Corollary 1 and its proof] and [9, Appendix B]. □

4.1 First method of the approximation

In (19), let us replace $K^*(\varepsilon)$ with $K_0(\varepsilon)$. Due to this replacement, we obtain the value

$$\begin{aligned} J_{\varepsilon 0,1}(z_0) &\triangleq z_0^T K_0(\varepsilon) z_0 \\ &= z_{0,1}^T K_{10} z_{0,1} + \varepsilon\big(2z_{0,1}^T K_{20} z_{0,2} + z_{0,2}^T K_{40} z_{0,2}\big) + \varepsilon^2\big(2z_{0,2}^T K_{50} z_{0,3} + z_{0,3}^T K_{60} z_{0,3}\big), \end{aligned} \tag{57}$$

where

$$z_{0,1} \in \mathbb{R}^{n-r}, \quad z_{0,2} \in \mathbb{R}^{r_1}, \quad z_{0,3} \in \mathbb{R}^{r-r_1}, \quad \mathrm{col}(z_{0,1}, z_{0,2}, z_{0,3}) = z_0. \tag{58}$$

Taking into account Proposition 2 (the equation (53)), the expression (57) can be rewritten as

$$J_{\varepsilon 0,1}(z_0) = \widetilde{J}_{\rm r}^*(z_{0,1}) + \varepsilon\big(2z_{0,1}^T K_{20} z_{0,2} + z_{0,2}^T K_{40} z_{0,2}\big) + \varepsilon^2\big(2z_{0,2}^T K_{50} z_{0,3} + z_{0,3}^T K_{60} z_{0,3}\big). \tag{59}$$

Theorem 1. *Let the assumptions (A2)–(A4) be valid. Then, the following inequality is satisfied:*

$$\big|J_\varepsilon^*(z_0) - J_{\varepsilon 0,1}(z_0)\big| \leq a\varepsilon\phi(z_0, \varepsilon), \quad \varepsilon \in (0, \varepsilon_1^*],$$

where the positive number a *is introduced in Lemma 2,*

$$\begin{aligned} \phi(z_0, \varepsilon) \triangleq \|z_{0,1}\|^2 &+ \varepsilon(2\|z_{0,1}\|\|z_{0,2}\| + 2\|z_{0,1}\|\|z_{0,3}\| + \|z_{0,2}\|^2) \\ &+ \varepsilon^2(2\|z_{0,2}\|\|z_{0,3}\| + \|z_{0,3}\|^2). \end{aligned} \tag{60}$$

Proof. Using Corollary 1 and the equations (46) and (58), we obtain the following expression for the CCIHG value:

$$\begin{aligned} J_\varepsilon^*(z_0) = z_{0,1}^T K_1^*(\varepsilon) z_{0,1} &+ \varepsilon\big(2z_{0,1}^T K_2^*(\varepsilon) z_{0,2} + 2z_{0,1}^T K_3^*(\varepsilon) z_{0,3} + z_{0,2}^T K_4^*(\varepsilon) z_{0,2}\big) \\ &+ \varepsilon^2\big(2z_{0,2}^T K_5^*(\varepsilon) z_{0,3} + z_{0,3}^T K_6^*(\varepsilon) z_{0,3}\big), \quad \varepsilon \in (0, \varepsilon^*]. \end{aligned} \tag{61}$$

Now, the statement of the theorem directly follows from the equations (37), (57), and (61), and Lemma 2 (the inequality (45)). □

Corollary 2. *Let the assumptions (A2)–(A4) be valid. Then, the following inequality is satisfied:*

$$\big|J_\varepsilon^*(z_0) - \widetilde{J}_{\rm r}^*(z_{0,1})\big| \leq \varepsilon\big(a\phi(z_0, \varepsilon) + \phi_1(z_0, \varepsilon)\big), \quad \varepsilon \in (0, \varepsilon_1^*],$$

where

$$\begin{aligned} \phi_1(z_0, \varepsilon) \triangleq 2\|K_{20}\|\|z_{0,1}\|\|z_{0,2}\| &+ \|K_{40}\|\|z_{0,2}\|^2 \\ &+ \varepsilon(2\|K_{50}\|\|z_{0,2}\|\|z_{0,3}\| + \|K_{60}\|\|z_{0,3}\|^2). \end{aligned}$$

Proof. The statement of the corollary directly follows from Theorem 1 and the equation (59). □

The following two assertions directly follows from Theorem 1.

Corollary 3. *Let the assumptions (A2)–(A4) be valid. Let $z_{0,1} = 0$, while $z_{0,2} \neq 0$. Then, the following inequality is satisfied:*

$$\left|J^*_\varepsilon(z_0) - J_{\varepsilon 0,1}(z_0)\right| \leq a\varepsilon^2 \phi_2(z_{0,2}, z_{0,3}, \varepsilon), \qquad \varepsilon \in (0, \varepsilon^*_1],$$

where

$$\phi_2(z_{0,2}, z_{0,3}, \varepsilon) \stackrel{\triangle}{=} \|z_{0,2}\|^2 + \varepsilon(2\|z_{0,2}\|\|z_{0,3}\| + \|z_{0,3}\|^2).$$

Corollary 4. *Let the assumptions (A2)–(A4) be valid. Let $z_{0,1} = 0$ and $z_{0,2} = 0$, while $z_{0,3} \neq 0$. Then, the following inequality is satisfied:*

$$\left|J^*_\varepsilon(z_0) - J_{\varepsilon 0,1}(z_0)\right| \leq a\varepsilon^3 \|z_{0,3}\|^2, \qquad \varepsilon \in (0, \varepsilon^*_1].$$

4.2 Second method of the approximation

In (18), we replace $K^*(\varepsilon)$ with $K_0(\varepsilon)$. This replacement yields the following new controls of the minimizer and the maximizer in the CCIHG:

$$u_{\varepsilon 0}(z) = -\varepsilon^{-2} B_{\mathrm{u}}^T K_0(\varepsilon) z, \qquad v_{\varepsilon 0}(z) = G_{\mathrm{v}}^{-1} B_{\mathrm{v}}^T K_0(\varepsilon) z.$$

Since $u_{\varepsilon 0}(z)$ and $v_{\varepsilon 0}(z)$ are linear with respect to z for all $\varepsilon \in (0, \varepsilon^*_1]$, then $\big(u_{\varepsilon 0}(z), v_{\varepsilon 0}(z)\big) \in \big(UV\big)$.

Employing this pair of controls in the CCIHG yields the value of the cost functional (3) $J_{\varepsilon 0,2}(z_0) \stackrel{\triangle}{=} J\big(u_{\varepsilon 0}(z,t), v_{\varepsilon 0}(z,t)\big)$ as:

$$J_{\varepsilon 0,2}(z_0) = \int_0^{+\infty} z_{\varepsilon 0}^T(t) \mathcal{D}_0(\varepsilon) z_{\varepsilon 0}(t) dt, \qquad \mathcal{D}_0(\varepsilon) \stackrel{\triangle}{=} D + K_0(\varepsilon)\big(S_{\mathrm{u}}(\varepsilon) - S_{\mathrm{v}}\big) K_0(\varepsilon), \tag{62}$$

where $z_{\varepsilon 0}(t)$ satisfies the initial-value problem

$$\begin{aligned} \dot{z}_{\varepsilon 0} = \Theta(\varepsilon) z_{\varepsilon 0}, \quad t \in [0, +\infty), \quad z_{\varepsilon 0}(0) = z_0, \\ \Theta(\varepsilon) \stackrel{\triangle}{=} A - \big(S_{\mathrm{u}}(\varepsilon) - S_{\mathrm{v}}\big) K_0(\varepsilon). \end{aligned} \tag{63}$$

Using the matrices $\Theta(\varepsilon)$ and $\mathcal{D}_0(\varepsilon)$, we construct the following Lyapunov equation with respect to the matrix $\mathcal{L}_0(\varepsilon)$:

$$\mathcal{L}_0(\varepsilon)\Theta(\varepsilon) + \Theta^T(\varepsilon)\mathcal{L}_0(\varepsilon) = -\mathcal{D}_0(\varepsilon). \tag{64}$$

Lemma 4. *Let the assumptions (A2)–(A4) be valid. Then, for all $\varepsilon \in (0, \varepsilon^*_1]$, the value $J_{\varepsilon 0,2}$, given by (62)–(63), can be represented in the form*

$$J_{\varepsilon 0,2}(z_0) = z_0^T \mathcal{L}_0(\varepsilon) z_0. \tag{65}$$

Proof. First of all, let us note that, due to Lemma 3, the matrix $\Theta(\varepsilon)$ is a Hurwitz matrix for all $\varepsilon \in (0, \varepsilon_1^*]$.

Solving the equation (63), we obtain

$$z_{\varepsilon 0}(t) = \exp\big(\Theta(\varepsilon)t\big)z_0, \quad t \in [0, +\infty), \quad \varepsilon \in (0, \varepsilon_1^*]. \tag{66}$$

Substitution of (66) into (62) yields

$$J_{\varepsilon 0,2}(z_0) = z_0^T \int_0^{+\infty} \exp\big(\Theta^T(\varepsilon)t\big)\mathcal{D}_0(\varepsilon)\exp\big(\Theta(\varepsilon)t\big)dt z_0, \quad \varepsilon \in (0, \varepsilon_1^*]. \tag{67}$$

Since the matrix $\Theta(\varepsilon)$ is a Hurwitz matrix, the integral in (67) converges.

Using the aforementioned property of the matrix $\Theta(\varepsilon)$ and the results of [4], we obtain that the Lyapunov equation (64) has the unique solution

$$\mathcal{L}_0(\varepsilon) = \int_0^{+\infty} \exp\big(\Theta^T(\varepsilon)t\big)\mathcal{D}_0(\varepsilon)\exp\big(\Theta(\varepsilon)t\big)dt, \quad \varepsilon \in (0, \varepsilon_1^*]. \tag{68}$$

Moreover, due to the symmetry of the matrix $\mathcal{D}_0(\varepsilon)$, the matrix $\mathcal{L}_0(\varepsilon)$ is symmetric.

Now, the comparison of the equalities (67) and (68) directly yields the equality (65), which completes the proof of the lemma. □

Using the symmetry of the matrix $\mathcal{L}_0(\varepsilon)$, we represent it in the block form similar to (20)

$$\mathcal{L}_0(\varepsilon) = \begin{bmatrix} \mathcal{L}_{10}(\varepsilon) & \varepsilon\mathcal{L}_{20}(\varepsilon) & \varepsilon\mathcal{L}_{30}(\varepsilon) \\ \varepsilon\mathcal{L}_{20}^T(\varepsilon) & \varepsilon\mathcal{L}_{40}(\varepsilon) & \varepsilon^2\mathcal{L}_{50}(\varepsilon) \\ \varepsilon\mathcal{L}_{30}^T(\varepsilon) & \varepsilon^2\mathcal{L}_{50}^T(\varepsilon) & \varepsilon^2\mathcal{L}_{60}(\varepsilon) \end{bmatrix}. \tag{69}$$

Lemma 5. *Let the assumptions (A2)–(A4) be valid. Then, there exists a positive number $\varepsilon_2^* \leq \varepsilon_1^*$ such that, for all $\varepsilon \in (0, \varepsilon_2^*]$, the following inequalities are satisfied:*

$$\big\|K_l^*(\varepsilon) - \mathcal{L}_{l0}(\varepsilon)\big\| \leq c\varepsilon^2, \quad l = 1, 2, ..., 6, \tag{70}$$

where $K_l^*(\varepsilon)$, $(l = 1, 2, ..., 6)$, are the components of the solution to the set (24)–(29) mentioned in Lemma 2; $c > 0$ is some constant independent of ε.

Proof. Consider the matrices

$$\Delta K_{\mathcal{L}}(\varepsilon) \stackrel{\triangle}{=} K^*(\varepsilon) - \mathcal{L}_0(\varepsilon), \quad \varepsilon \in (0, \varepsilon_1^*], \tag{71}$$

$$\Delta K_0(\varepsilon) \stackrel{\triangle}{=} K^*(\varepsilon) - K_0(\varepsilon), \quad \varepsilon \in (0, \varepsilon_1^*], \tag{72}$$

where the matrices $K^*(\varepsilon)$ and $K_0(\varepsilon)$ are given by (46) and (54), respectively.

Taking into account that $K^*(\varepsilon)$ satisfies the equation (15) and $\mathcal{L}_0(\varepsilon)$ satisfies the equation (64), we obtain that $\Delta K_{\mathcal{L}}(\varepsilon)$ satisfies the following equation:

$$\Delta K_{\mathcal{L}}(\varepsilon)\Theta(\varepsilon) + \Theta^T(\varepsilon)\Delta K_{\mathcal{L}}(\varepsilon) = \Delta K_0(\varepsilon)\big(S_{\mathrm{u}}(\varepsilon) - S_{\mathrm{v}}\big)\Delta K_0(\varepsilon), \quad \varepsilon \in (0, \varepsilon_1^*].$$

Solving this equation, we obtain

$$\Delta K_{\mathcal{L}}(\varepsilon) = -\int_0^{+\infty} \exp\big(\Theta^T(\varepsilon)t\big)\Delta K_0(\varepsilon)\big(S_{\mathrm{u}}(\varepsilon) - S_{\mathrm{v}}\big)\Delta K_0(\varepsilon)\exp\big(\Theta(\varepsilon)t\big)dt, \quad \varepsilon \in (0, \varepsilon_1^*]. \tag{73}$$

Let us estimate the matrix-valued function $\Psi(t,\varepsilon) \stackrel{\triangle}{=} \exp\big(\Theta(\varepsilon)t\big)$. To obtain this estimate, we partition $\Psi(t,\varepsilon)$ into blocks as:

$$\Psi(t,\varepsilon) = \left[\begin{array}{ccc} \Psi_1(t,\varepsilon) & \Psi_2(t,\varepsilon) & \Psi_3(t,\varepsilon) \\ \Psi_4(t,\varepsilon) & \Psi_5(t,\varepsilon) & \Psi_6(t,\varepsilon) \\ \Psi_7(t,\varepsilon) & \Psi_8(t,\varepsilon) & \Psi_9(t,\varepsilon) \end{array}\right], \tag{74}$$

where the blocks are of the same dimensions as the corresponding blocks in (20).

By virtue of the results of [9] (Appendix B), we have the following estimates of these blocks:

$$\begin{aligned} \big\|\Psi_{\alpha_1}(t,\varepsilon)\big\| &\le c_1\exp(-\beta_1 t), \qquad \alpha_1 = 1,4,7,9, \\ \big\|\Psi_{\alpha_2}(t,\varepsilon)\big\| &\le c_1\varepsilon\exp(-\beta_1 t), \qquad \alpha_2 = 2,8, \\ \big\|\Psi_{\alpha_3}(t,\varepsilon)\big\| &\le c_1\varepsilon^2\exp(-\beta_1 t), \qquad \alpha_3 = 3,6, \\ \big\|\Psi_5(t,\varepsilon)\big\| &\le c_1\Big(\varepsilon\exp(-\beta_1 t) + \exp\big(-\beta_2 t/\varepsilon\big)\Big), \\ & t \ge 0, \qquad \varepsilon \in (0,\varepsilon_2^*], \end{aligned} \tag{75}$$

where $0 < \varepsilon_2^* \le \varepsilon_1^*$ is some sufficiently small number; c_1, β_1 and β_2 are some positive numbers independent of ε.

Now, using the block forms of the matrices $S_{\rm u}(\varepsilon)$, $S_{\rm v}$, $K^*(\varepsilon)$, $K_0(\varepsilon)$, $\mathcal{L}_0(\varepsilon)$, $\Psi(t,\varepsilon)$ (see the equations (16), (22), (46), (54), (69), and (74)), as well as Lemma 2, the equations (71), (72), and (73) and the estimates (75), we obtain by a routine algebra the validity of the inequalities (70). This completes the proof of the lemma. □

Theorem 2. *Let the assumptions (A2)–(A4) be valid. Then, the following inequality is valid:*

$$\big|J_\varepsilon^*(z_0) - J_{\varepsilon 0,2}(z_0)\big| \le c\varepsilon^2\phi(z_0,\varepsilon), \qquad \varepsilon \in (0,\varepsilon_2^*],$$

where the positive numbers ε_2^* and c are introduced in Lemma 5, the value $\phi(z_0,\varepsilon)$ is given by (58), (60).

Proof. Substituting (69) into (65) and using (58) yield after a routine rearrangement the following expression for $J_{\varepsilon 0,2}(z_0)$:

$$\begin{aligned} J_{\varepsilon 0,2}(z_0) = z_{0,1}^T\mathcal{L}_{10}(\varepsilon)z_{0,1} + \varepsilon\big(2z_{0,1}^T\mathcal{L}_{20}(\varepsilon)z_{0,2} + 2z_{0,1}^T\mathcal{L}_{30}(\varepsilon)z_{0,3} + z_{0,2}^T\mathcal{L}_{40}(\varepsilon)z_{0,2}\big) \\ +\varepsilon^2\big(2z_{0,2}^T\mathcal{L}_{50}(\varepsilon)z_{0,3} + z_{0,3}^T\mathcal{L}_{60}(\varepsilon)z_{0,3}\big). \end{aligned} \tag{76}$$

Now, the statement of the theorem directly follows from the equations (61) and (76), Lemma 5 and the inequality $\varepsilon_2^* \le \varepsilon^*$. □

5 Examples

5.1 Example 1

In the example, presented in this subsection, the theoretical results of the previous sections, are illustrated by numerical/computer simulation.

Consider a particular case of the CCIHG with the following data:

$$n = 3, \quad r = s = 2,$$

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_{\rm v} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G_{\rm v} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \tag{77}$$

yielding

$$D_1 = 0.5, \quad D_2 = \begin{bmatrix} 1.2 & 0 \\ 0 & 0 \end{bmatrix}, \quad r_1 = 1, \qquad S_{\rm v} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{78}$$

Due to (2) and (16),

$$B_{\rm u} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_{\rm u}(\varepsilon) = \frac{1}{\varepsilon^2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{79}$$

5.1.1 Verification of assumptions (A1) – (A4)

Let $K^*(\varepsilon)$ be the solution of the matrix Riccati algebraic equation (15) for the matrix-valued coefficients given in (77)–(79). In this example, for the sake of the comparison with the zero-order asymptotic solution, $K^*(\varepsilon)$ was calculated numerically using the MATLAB function `icare(A,[],D,[],[],[],-(Su-Sw))`. Numerical simulation shows that the equation (15) with the data (78) admits the unique symmetric stabilizing solution for $0 < \varepsilon < \widehat{\varepsilon} = 1.205$.

First, we check that assumption (**A1**) holds.

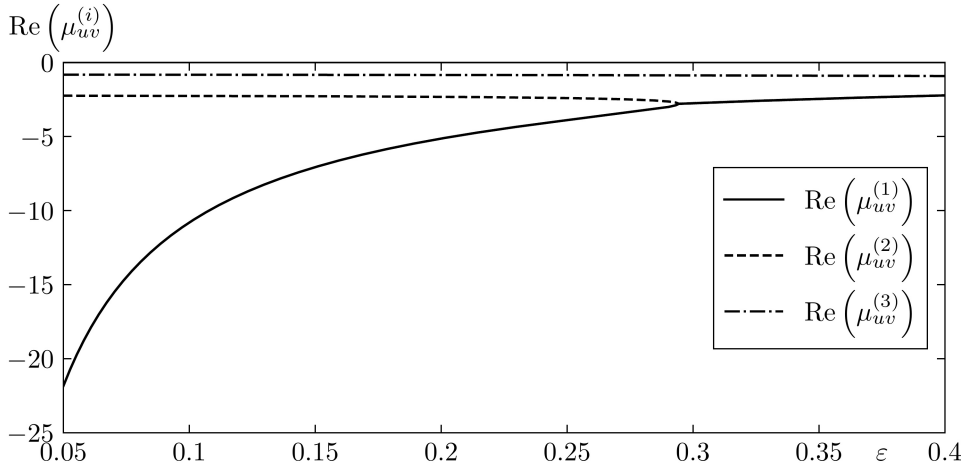


Figure 1: Real parts of $\mu_{uv}^{(i)}(\varepsilon)$, $i = 1, 2, 3$

In Figure 1, the real parts of the eigenvalues $\mu_{uv}^{(i)}(\varepsilon)$ $(i = 1, 2, 3)$ of the game closed-loop matrix

$$A_{uv}(\varepsilon) = A - (S_u(\varepsilon) - S_v)K^*(\varepsilon),$$

are depicted (for the sake of demonstration, for $\varepsilon \in [0.05, 0.4] \subset (0, \widehat{\varepsilon})$). The real parts of the eigenvalues $\mu_u^{(i)}(\varepsilon)$ $(i = 1, 2, 3)$ the minimizer's closed-loop matrix

$$A_u(\varepsilon) = A - S_u(\varepsilon)K^*(\varepsilon).$$

are shown in Fig. 2.

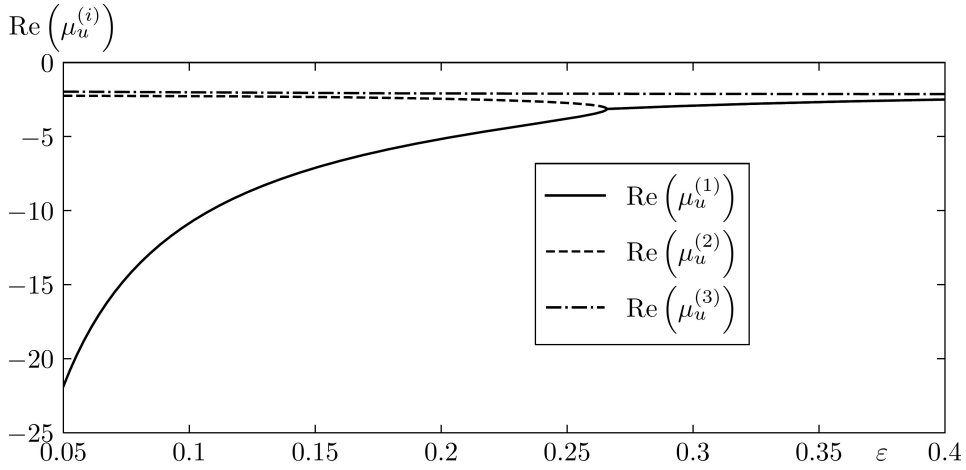


Figure 2: Real parts of $\mu_u^{(i)}(\varepsilon)$, $i = 1, 2, 3$

It is seen that $\mathrm{Re}\left(\mu_{uv}^{(i)}(\varepsilon)\right) < 0$, $\mathrm{Re}\left(\mu_u^{(i)}(\varepsilon)\right) < 0$, $i = 1, 2, 3$, for all ε in the shown interval, i.e., the MATLAB function `icare` returns the symmetric stabilizing solution of (15), (77)–(79), meaning that the assumption (**A1**) is satisfied. Thus, this numerical solution $K^*(\varepsilon)$ is suitable for the comparison with the zero-order asymptotic approximation. One can observe that $\mathrm{Re}\big(\mu_{uv}^{(1)}(\varepsilon)\big) = \mathrm{Re}\big(\mu_{uv}^{(2)}(\varepsilon)\big)$ for $\varepsilon > 0.294$, and $\mathrm{Re}\big(\mu_u^{(1)}(\varepsilon)\big) = \mathrm{Re}\big(\mu_u^{(2)}(\varepsilon)\big)$ for $\varepsilon > 0.266$.

The assumption (**A2**) is satisfied by choosing the matrix A in (77): $\bar{A}_3 = A_{13} = 0$.

Now let us check the assumption (**A3**). To this end, the inequalities

$$\kappa_{uv}(K_{10}) = \bar{A}_1 - \left(\frac{\bar{A}_2^2}{\Lambda_1} - \bar{S}_{\mathrm{v}1}\right) K_{10} < 0, \tag{80}$$

$$\kappa_u(K_{10}) = \bar{A}_1 - \frac{\bar{A}_2^2 K_{10}}{\Lambda_1} < 0, \tag{81}$$

should be satisfied. In this example, $\bar{A}_1 = A_{11} = -1$, $\bar{A}_2 = A_{12} = 2$, $\bar{S}_{\mathrm{v}1} = S_{\mathrm{v}_{11}} = 4$, $\Lambda_1 = D_{2_{11}} = 1.2$, and the equation (40) is the scalar quadratic equation

$$\frac{2}{3}K_{10}^2 - 2K_{10} + \frac{1}{2} = 0,$$

admitting the solutions $K_{10}^{(1)} = \dfrac{3+\sqrt{6}}{2}$, $K_{10}^{(2)} = \dfrac{3-\sqrt{6}}{2}$. By calculating the left-hand sides of the inequalities (80)–(81), $\kappa_{uv}(K_{10}^{(1)}) = 0.8165$, $\kappa_u(K_{10}^{(1)}) = -10.0825$, $\kappa_{uv}(K_{10}^{(2)}) = -0.8165$, $\kappa_u(K_{10}^{(2)}) = -1.9175$, i.e., the inequalities (80)–(81) are satisfied (and, consequently, (**A3**) holds) for

$$K_{10} = K_{10}^{(2)} = 0.2753. \tag{82}$$

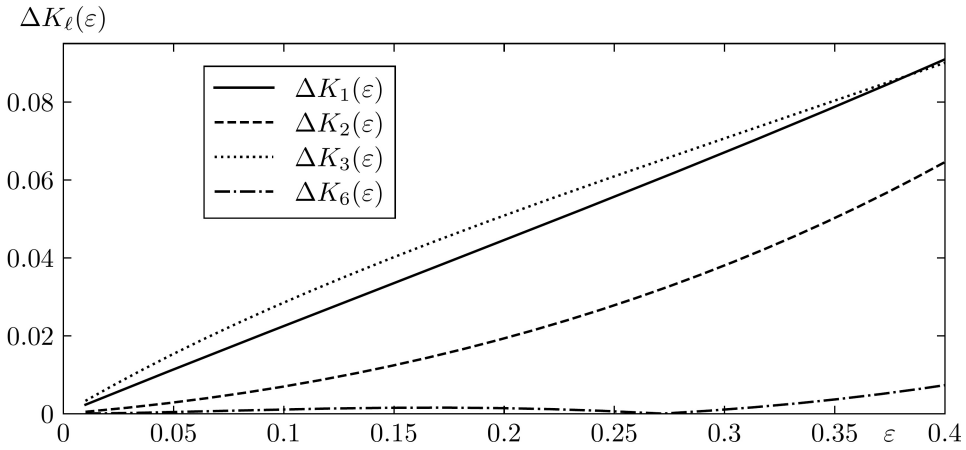


Figure 3: Matrix approximation errors $\Delta K_\ell(\varepsilon)$, $\ell = 1, 2, 3, 6$

In this example, $\bar{A}_6 = A_{23} = 1$, $\bar{A}_9 = A_{33} = -2$, $r = 2$, $r_1 = 1$, and

$$\operatorname{rank}\left[\bar{A}_6^T, \bar{A}_9^T \bar{A}_6^T, ..., \left(\bar{A}_9^T\right)^{r-r_1-1} \bar{A}_6^T\right] = \operatorname{rank}\left[\bar{A}_6\right] = \operatorname{rank}\left[1\right] = 1 = r - r_1,$$

i.e., the assumption (**A4**) is satisfied.

5.1.2 Zero-order matrix approximation

Due to (36)–(39),

$$K_{20} = \frac{3 - \sqrt{6}}{\sqrt{1.2}} = 1.4476, \quad K_{30} = 0, \quad K_{40} = \sqrt{1.2} = 1.0954, \quad K_{50} = 1. \tag{83}$$

The equation (41) is the scalar quadratic equation

$$-K_{60}^2 - 4K_{60} + 1 = 0,$$

admitting the unique positive solution

$$K_{60} = \sqrt{5} - 2 = 0.2361. \tag{84}$$

In Figures 3–4, the approximation errors

$$\Delta K_\ell(\varepsilon) = \left|K_\ell^*(\varepsilon) - K_{\ell 0}\right|,$$

are depicted, for the sake of demonstration, for $\varepsilon \in [0.01, 0.4]$, illustrating Lemma 2.

Due to (82)–(83) and (84), the zero-order approximation of the matrix $K^*(\varepsilon)$ is

$$K_0(\varepsilon) = \begin{bmatrix} 0.2753 & 1.4476\varepsilon & 0 \\ 1.4476\varepsilon & 1.0954\varepsilon & \varepsilon^2 \\ 0 & \varepsilon^2 & 0.2361\varepsilon^2 \end{bmatrix}.$$

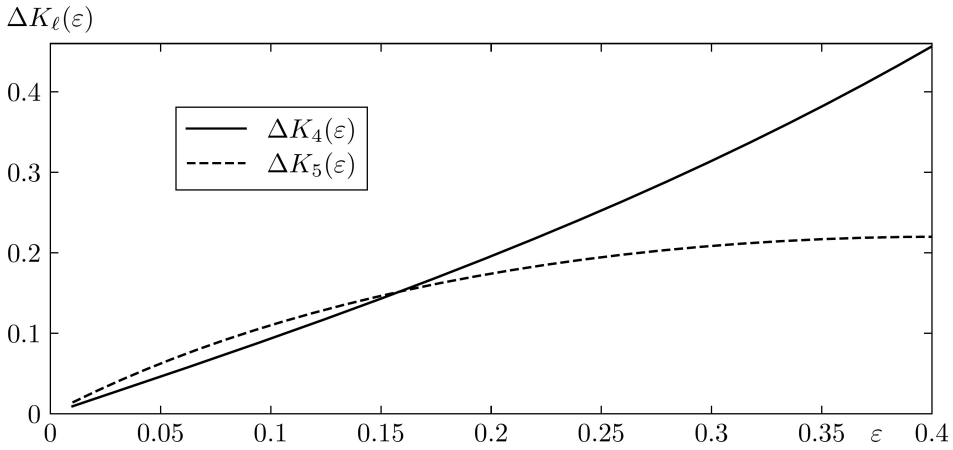


Figure 4: Matrix approximation errors $\Delta K_\ell(\varepsilon)$, $\ell = 4, 5$

In Figure 5, the real parts of the eigenvalues $\nu_{uv}^{(i)}(\varepsilon)$ $(i = 1, 2, 3)$ of the game closed-loop matrix

$$\widehat{A}_{uv}(\varepsilon) = A - \big(S_{\rm u}(\varepsilon) - S_{\rm v}\big)K_0(\varepsilon),$$

are depicted (for the sake of demonstration, for $\varepsilon \in [0.05, 0.4]$). The real parts of the eigenvalues $\nu_u^{(i)}(\varepsilon)$ $(i = 1, 2, 3)$ the minimizer's closed-loop matrix

$$\widehat{A}_u(\varepsilon) = A - S_{\rm u}(\varepsilon)K_0(\varepsilon).$$

are shown in Fig. 6.

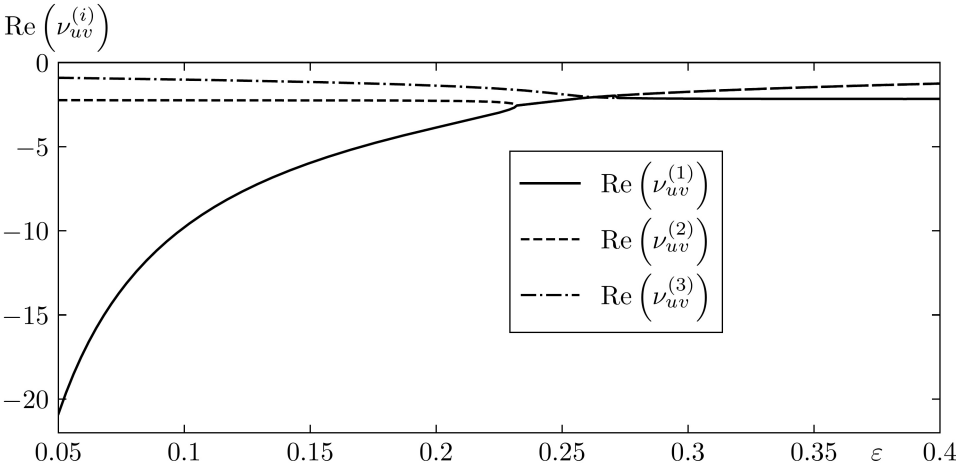


Figure 5: Real parts of $\nu_{uv}^{(i)}(\varepsilon)$, $i = 1, 2, 3$

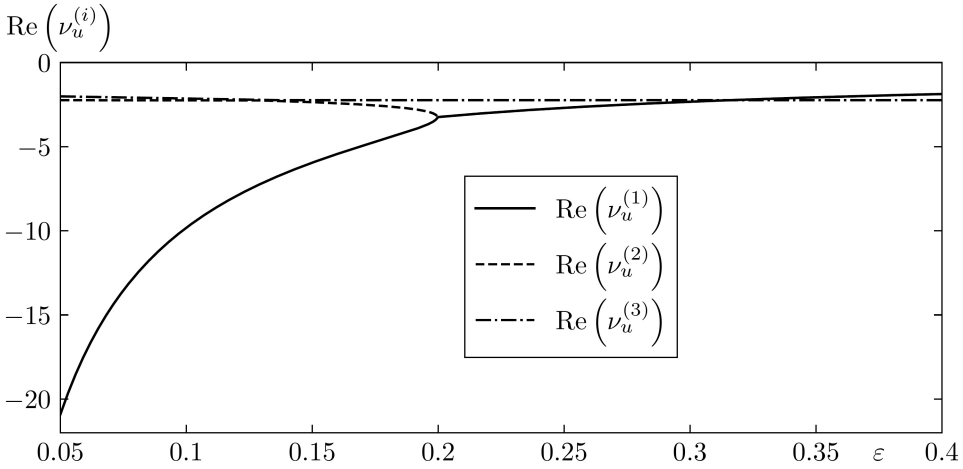


Figure 6: Real parts of $\nu_u^{(i)}(\varepsilon)$, $i = 1, 2, 3$

It is seen that $\mathrm{Re}\left(\nu_{uv}^{(i)}(\varepsilon)\right) < 0$, $\mathrm{Re}\left(\nu_u^{(i)}(\varepsilon)\right) < 0$, $i = 1, 2, 3$, for all ε in the shown interval, i.e., the approximating matrix satisfies (**A1**) with replacing $K^*(\varepsilon)$ by $K_0(\varepsilon)$. One can observe that $\mathrm{Re}\left(\nu_{uv}^{(1)}(\varepsilon)\right) = \mathrm{Re}\left(\nu_{uv}^{(2)}(\varepsilon)\right)$ for $\varepsilon > 0.232$, and $\mathrm{Re}\left(\nu_u^{(1)}(\varepsilon)\right) = \mathrm{Re}\left(\nu_u^{(2)}(\varepsilon)\right)$ for $\varepsilon > 0.2$.

5.1.3 Game value approximation

In Figure 7, the absolute errors

$$\Delta J^{(1)}(\varepsilon, z_0) = \left|J_\varepsilon^*(z_0) - J_{\varepsilon 0,1}(z_0)\right| = \left|z_0^T K^*(\varepsilon) z_0 - z_0^T K_0(\varepsilon) z_0\right|, \tag{85}$$

are depicted for $\varepsilon \in [0.01, 0.4]$ and for three initial positions $z_0 = z_0^{(1)} = [1, 2, 3]^T$, $z_0 = z_0^{(2)} = [0, 2, 3]^T$, and $z_0 = z_0^{(3)} = [0, 0, 3]^T$.
It is seen that the absolute errors of the first game value approximation are smaller with increasing of the number of first zero components. This illustrates Theorem 1 and Corollaries 3–4.

The respective relative errors

$$\delta J^{(1)}(\varepsilon, z_0) = \frac{\Delta J^{(1)}(\varepsilon)}{J_\varepsilon^*(z_0)} \cdot 100\%,$$

are depicted in Fig. 8.

The first approximation errors for $\varepsilon = 0.01, 0.05, 0.1, 0.2, 0.4$ are presented in Table 1.
In Figure 9, the absolute errors

$$\Delta J^{(2)}(\varepsilon, z_0) = \left|J_\varepsilon^*(z_0) - J_{\varepsilon 0,2}(z_0)\right| = \left|z_0^T K^*(\varepsilon) z_0 - z_0^T \mathcal{L}(\varepsilon) z_0\right|,$$

are depicted for $\varepsilon \in [0.01, 0.4]$ and for three initial positions $z_0 = z_0^{(1)} = [1, 2, 3]^T$, $z_0 = z_0^{(2)} = [0, 2, 3]^T$, and $z_0 = z_0^{(3)} = [0, 0, 3]^T$. In comparison with Fig. 7, it is seen that the absolute errors of the second game value approximation are considerably smaller than the errors (85) of the first approximation. This illustrates Theorem 2.

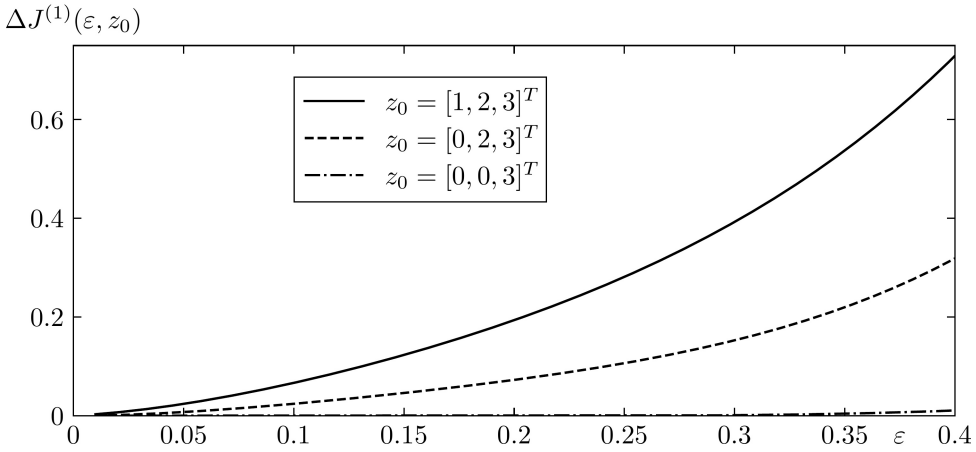


Figure 7: Absolute errors of the first game value approximation

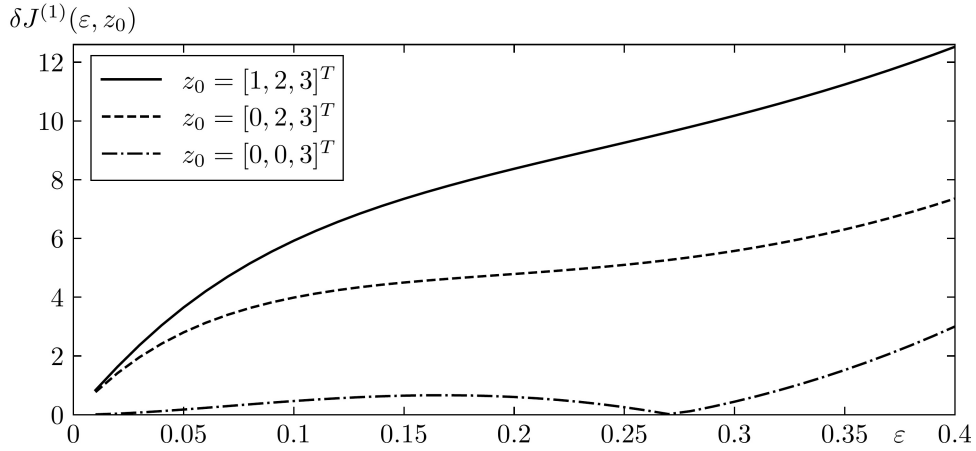


Figure 8: Relative errors of the first game value approximation

The matrices $\mathcal{L}(\varepsilon)$ defined as the solutions of the Lyapunov matrix equation (64), were calculated numerically using the MATLAB function `lyap(Theta,D0)`. The respective relative errors

$$\delta J^{(2)}(\varepsilon, z_0) = \frac{\Delta J^{(2)}(\varepsilon)}{J^*_\varepsilon(z_0)} \cdot 100\%,$$

are depicted in Fig. 10. These errors are very small, showing the high accuracy of the second approximation.

Table 1: Errors of game value $J^*_\varepsilon(z_0)$ for the first method of approximation

	$z_0 = z_0^{(1)}$		$z_0 = z_0^{(2)}$		$z_0 = z_0^{(3)}$	
ε	$\Delta J^{(1)}$	$\delta J^{(1)}$, %	$\Delta J^{(1)}$	$\delta J^{(1)}$, %	$\Delta J^{(1)}$	$\delta J^{(1)}$, %
0.01	0.0029	0.8344	$3.5033 \cdot 10^{-4}$	0.7686	$2.0255 \cdot 10^{-8}$	0.0095
0.05	0.0239	3.6521	0.0073	2.8025	$9.2815 \cdot 10^{-6}$	0.175
0.1	0.0665	5.9229	0.0241	3.9869	$9.8263 \cdot 10^{-5}$	0.4646
0.2	0.1936	8.3715	0.0725	4.7894	$5.0877 \cdot 10^{-4}$	0.6023
0.4	0.729	12.5236	0.3186	7.3565	0.0105	3.0016

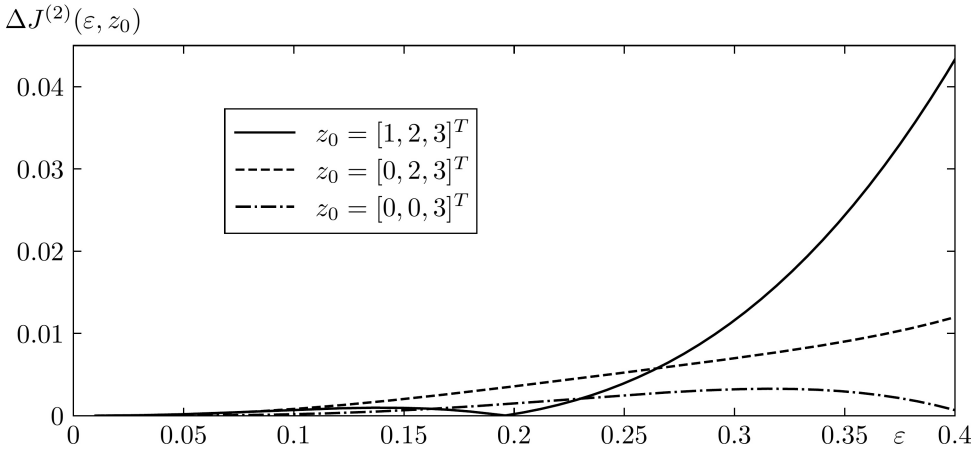


Figure 9: Absolute errors of the second game value approximation

The second approximation errors for $\varepsilon = 0.01, 0.05, 0.1, 0.2, 0.4$ are presented in Table 2.

Table 2: Errors of game value $J^*_\varepsilon(z_0)$ for the second method of approximation

	$z_0 = z_0^{(1)}$		$z_0 = z_0^{(2)}$		$z_0 = z_0^{(3)}$	
ε	$\Delta J^{(2)}$	$\delta J^{(2)}$, %	$\Delta J^{(2)}$	$\delta J^{(2)}$, %	$\Delta J^{(2)}$	$\delta J^{(2)}$, %
0.01	$3.9254 \cdot 10^{-6}$	0.0011	$1.4381 \cdot 10^{-6}$	0.0032	$3.0035 \cdot 10^{-8}$	0.0141
0.05	$1.7833 \cdot 10^{-4}$	0.0273	$1.3877 \cdot 10^{-4}$	0.053	$1.4758 \cdot 10^{-5}$	0.2783
0.1	$7.0869 \cdot 10^{-4}$	0.0632	$8.1292 \cdot 10^{-4}$	0.1347	$1.7522 \cdot 10^{-4}$	0.8285
0.2	$1.9464 \cdot 10^{-4}$	0.0084	0.0036	0.2358	0.0015	1.755
0.4	0.0433	0.7437	0.012	0.2765	$6.7258 \cdot 10^{-4}$	0.1919

Let us compare the CPU time required for the first and the second approximation. The first approximation method average CPU time for $z_0 = z_0^{(1)}$ and for $\varepsilon = 0.01, 0.05, 0.1, 0.2, 0.4$ is $1.5625 \cdot 10^{-5}$ sec., whereas the second approximation method average CPU time for the same parameters is $2.6563 \cdot 10^{-4}$ sec. Thus, the considerably more accurate second method requires more CPU time than the first method. However, both values are small and the difference is not significant.

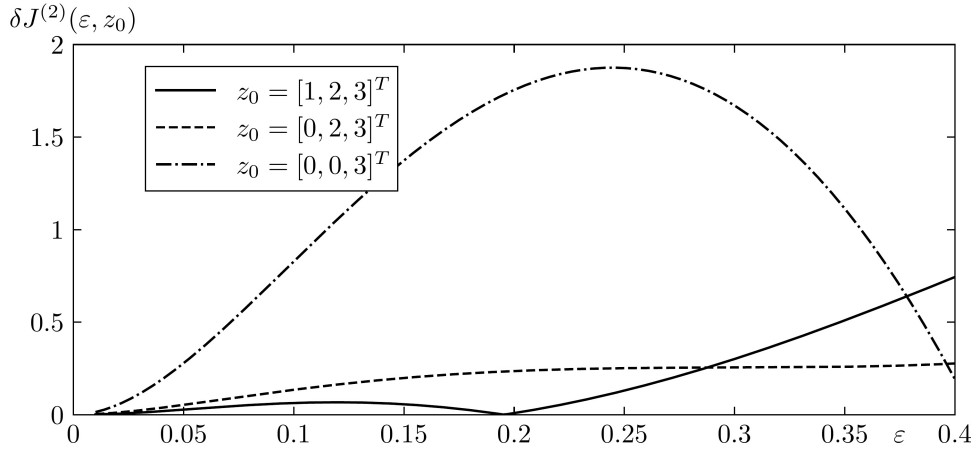


Figure 10: Relative errors of the second game value approximation

5.2 Example 2

In the example of this subsection, it is shown that the assumption (A2) is essential for the derivation of the asymptotic solution to the equation (15) subject to the block representation (20).

Namely, we consider the differential game (1), (3) and the corresponding matrix Riccati algebraic equation (15) for the same data as in (77)–(79), excepting the matrix A. In the present example, we choose the matrix A as:

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix},$$

meaning that $\bar{A}_3 = 1 \neq 0$, i.e., the assumption (A2) is not valid.

Using the aforementioned data and the the block representation (20) of solution to the equation (15), we can transform this equation into the following equivalent set of the equations:

$$-2K_1(\varepsilon) - K_2^2(\varepsilon) - K_3^2(\varepsilon) + 4K_1^2(\varepsilon) + 0.5 + 0.5\varepsilon^2 K_2^2(\varepsilon) = 0, \tag{86}$$

$$\begin{aligned} 2K_1(\varepsilon) - K_2(\varepsilon)K_4(\varepsilon) - \varepsilon K_2(\varepsilon) - \varepsilon K_3(\varepsilon)K_5(\varepsilon) \\ +4\varepsilon K_1(\varepsilon)K_2(\varepsilon) + 0.5\varepsilon^2 K_2(\varepsilon)K_4(\varepsilon) = 0, \end{aligned} \tag{87}$$

$$\begin{aligned} K_1(\varepsilon) + \varepsilon K_2(\varepsilon) - 3\varepsilon K_3(\varepsilon) - \varepsilon K_2(\varepsilon)K_5(\varepsilon) - \varepsilon K_3(\varepsilon)K_6(\varepsilon) \\ +4\varepsilon K_1(\varepsilon)K_3(\varepsilon) + 0.5\varepsilon^3 K_2(\varepsilon)K_5(\varepsilon) = 0, \end{aligned} \tag{88}$$

$$-K_4^2(\varepsilon) + 1.2 + 4\varepsilon K_2(\varepsilon) - \varepsilon^2 K_5^2(\varepsilon) + 4\varepsilon^2 K_2^2(\varepsilon) + 0.5\varepsilon^2 K_4^2(\varepsilon) = 0, \tag{89}$$

$$\begin{aligned} K_2(\varepsilon) + K_4(\varepsilon) + 2K_3(\varepsilon) - K_4(\varepsilon)K_5(\varepsilon) - 2\varepsilon K_5(\varepsilon) \\ -\varepsilon K_5(\varepsilon)K_6(\varepsilon) + 4\varepsilon K_2(\varepsilon)K_3(\varepsilon) + 0.5\varepsilon^2 K_4(\varepsilon)K_5(\varepsilon) = 0, \end{aligned} \tag{90}$$

$$2K_3(\varepsilon) + 2\varepsilon K_5(\varepsilon) - 4\varepsilon K_6(\varepsilon) - \varepsilon K_5^2(\varepsilon) - \varepsilon K_6^2(\varepsilon) + 4\varepsilon K_3^2(\varepsilon) + 0.5\varepsilon^3 K_5^2(\varepsilon) = 0. \tag{91}$$

Similarly to Section 3.2.2, we look for the zero-order asymptotic solution

$$\{K_{10}, K_{20}, K_{30}, K_{40}, K_{50}, K_{60}\}$$

of the set (86)–(91). Setting formally $\varepsilon = 0$ in this set, we obtain the equations for its zero-order asymptotic solution

$$-2K_{10} - K_{20}^2 - K_{30}^2 + 4K_{10}^2 + 0.5 = 0, \tag{92}$$

$$2K_{10} - K_{20}K_{40} = 0, \tag{93}$$

$$K_{10} = 0, \tag{94}$$

$$-K_{40}^2 + 1.2 = 0, \tag{95}$$

$$K_{20} + K_{40} + 2K_{30} - K_{40}K_{50} = 0, \tag{96}$$

$$2K_{30} = 0. \tag{97}$$

The equations (93)–(95) yield $K_{20} = 0$. The latter, along with the equations (92), (94), and (97), yields the contradiction $0.5 = 0$. Moreover, the set (92)–(97) does not yield the value K_{60}. These features of the set (92)–(97) mean that if the assumption (A2) is not fulfilled, the asymptotic solution to the equation (15) subject to the block representation (20) cannot be obtained, i.e., the assumption (A2) is essential in the obtaining of such asymptotic solution.

5.3 Example 3

In the example of this subsection, it is shown that the four-block-form representation of the matrix $K(\varepsilon)$, used in [3, 7, 14, 15, 16], is not applicable for the derivation of the asymptotic solution to the matrix Riccati algebraic equation (15) in the case of (13)–(14). We consider the differential game (1), (3) and the corresponding matrix Riccati algebraic equation (15) for the same data as in (77)–(79). Due to these data and the equations (5) and (16). we have

$$A_1 = -1, \quad A_2 = [2\ ,\ 0], \quad A_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix},$$
$$S_{\mathrm{u}1} = 0, \quad S_{\mathrm{u}2} = [0\ ,\ 0], \quad S_{\mathrm{u}3}(\varepsilon) = \frac{1}{\varepsilon^2} I_2,$$
$$S_{\mathrm{v}1} = 4, \quad S_{\mathrm{v}2} = [0\ ,\ 0], \quad S_{\mathrm{v}3} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}. \tag{98}$$

Due to [3, 7, 14, 15, 16], we look for the solution $K(\varepsilon)$ of the equation (15) in the following block form:

$$K(\varepsilon) = \begin{bmatrix} \widetilde{K}_1(\varepsilon) & \varepsilon\widetilde{K}_2(\varepsilon) \\ \varepsilon\widetilde{K}_2^T(\varepsilon) & \varepsilon\widetilde{K}_3(\varepsilon) \end{bmatrix}, \tag{99}$$

where $\widetilde{K}_1(\varepsilon)$ is scalar, $\widetilde{K}_2(\varepsilon)$ is of the dimension 1×2, $\widetilde{K}_3(\varepsilon)$ is of the dimension 2×2 and this matrix is symmetric.

Using the results of [7], the matrices D_1, D_2 (see the equation (78)) and the matrices in the equations (98) and (99), we can transform the equation (15) into the following equivalent set of the equations:

$$-2\widetilde{K}_1(\varepsilon)+4\widetilde{K}_1^2(\varepsilon)-\widetilde{K}_2(\varepsilon)\big(I_2-\varepsilon^2 S_{v3}\big)\widetilde{K}_2^T(\varepsilon)+0.5=0, \tag{100}$$

$$\begin{aligned}&\widetilde{K}_1(\varepsilon)[2\ ,\ 0]+\varepsilon\widetilde{K}_2(\varepsilon)\begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}-\varepsilon\widetilde{K}_2(\varepsilon)\\ &+4\varepsilon\widetilde{K}_1(\varepsilon)\widetilde{K}_2(\varepsilon)-\widetilde{K}_2(\varepsilon)\big(I_2-\varepsilon^2 S_{v3}\big)\widetilde{K}_3(\varepsilon)=0,\end{aligned} \tag{101}$$

$$\begin{aligned}&\varepsilon\widetilde{K}_2^T(\varepsilon)[2\ ,\ 0]+\varepsilon\widetilde{K}_3(\varepsilon)\begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}+\varepsilon\begin{bmatrix}2\\ 0\end{bmatrix}\widetilde{K}_2(\varepsilon)+\varepsilon\begin{bmatrix}0 & 0\\ 1 & -2\end{bmatrix}\widetilde{K}_3(\varepsilon)\\ &+4\varepsilon^2\widetilde{K}_2^T(\varepsilon)\widetilde{K}_2(\varepsilon)-\widetilde{K}_3(\varepsilon)\big(I_2-\varepsilon^2 S_{v3}\big)\widetilde{K}_3(\varepsilon)+\begin{bmatrix}1.2 & 0\\ 0 & 0\end{bmatrix}=0.\end{aligned} \tag{102}$$

We look for the zero-order asymptotic solution $\{\widetilde{K}_{10},\widetilde{K}_{20},\widetilde{K}_{30}\}$ of the set (100)–(102). Setting formally $\varepsilon=0$ in this set, we obtain the equations for its zero-order asymptotic solution

$$-2\widetilde{K}_{10}+4\widetilde{K}_{10}^2-\widetilde{K}_{20}\widetilde{K}_{20}^T+0.5=0, \tag{103}$$

$$\widetilde{K}_{10}[2\ ,\ 0]-\widetilde{K}_{20}\widetilde{K}_{30}=0, \tag{104}$$

$$-\widetilde{K}_{30}^2+\begin{bmatrix}1.2 & 0\\ 0 & 0\end{bmatrix}=0. \tag{105}$$

The unique symmetric positive semi-definite solution of the equation (105) is

$$\widetilde{K}_{30}=\begin{bmatrix}\sqrt{1.2} & 0\\ 0 & 0\end{bmatrix}. \tag{106}$$

Let us represent the unknown matrix $\widetilde{K}_{20}$ as $\widetilde{K}_{20}=[\widetilde{K}_{20,1}\ ,\ \widetilde{K}_{20,2}]$. Then, substitution of (106) into (104) yields after a routine algebra

$$[2\widetilde{K}_{10}\ ,\ 0]-[\sqrt{1.2}\widetilde{K}_{20,1}\ ,\ 0]=[0\ ,\ 0],$$

implying

$$\widetilde{K}_{20,1}=\frac{2}{\sqrt{1.2}}\widetilde{K}_{10}.$$

Substituting the latter into the equation (103), we obtain the equation

$$-2\widetilde{K}_{10}+4\widetilde{K}_{10}^2-\frac{4}{1.2}\widetilde{K}_{10}-\widetilde{K}_{20,2}^2+0.5=0. \tag{107}$$

This scalar equation is with respect to two scalar unknowns $\widetilde{K}_{10}$ and $\widetilde{K}_{20,2}$. Therefore, the equation (107) has infinitely many solutions, yielding infinitely many possible zero-order asymptotic solutions to the equation (15) in the case of (13)–(14). Hence, the four-block-form representation of the matrix $K(\varepsilon)$, used in [3, 7, 14, 15, 16], is not applicable for the derivation of the asymptotic solution to such a class of the equations.

6 Conclusions

In this paper, a two-player infinite-horizon zero-sum linear-quadratic differential game was studied in the case where the control cost of the minimizing player (the minimizer) is much smaller than the control cost of its opponent and the state cost. This smallness is due to the presence of the small multiplier (parameter) $\varepsilon > 0$ in the control cost of the minimizer. Thus, the considered game is a cheap control game. One more feature of the game is that its dynamics has slow and fast state variables, and the cost of the fast state in the integrand of the cost functional is a positive semi-definite (but non-zero) quadratic form. For this game, the state-feedback solution was considered. The new approach to the asymptotic solution of the game-theoretic matrix Riccati algebraic equation, associated with the considered game by the solvability conditions, was proposed. Using this approach, the zero-order asymptotic solution to the matrix Riccati algebraic equation was obtained. Based on this asymptotic solution, the ε-free conditions, providing the existence of the solution to the cheap control differential game for all sufficiently small values of ε, are established. Two methods of approximation of the game value were developed. The first method directly uses the aforementioned asymptotic solution and the expression for the game value. Namely, in this method the exact solution of the matrix Riccati algebraic equation, appearing in the expression for the game value, is replaced with the asymptotic solution of this equation. The second method yields the approximation of the game value as the outcome of the game, generated by the auxiliary pair of the players' controls. The latter is designed using the asymptotic solution of the matrix Riccati algebraic equation. In the numerical example, both methods were compared with each other by computer simulation. This simulation shows that the second method, although requiring more CPU time, is much more accurate than the first one. The second method provides a very accurate approximation of the game value even for not too small values of ε.

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