



First-Order Tableaux with Unification for Henkin Quantifiers

Matthias Baaz^{1*}, Mariami Gamsakhurdia¹, Anela Lolić^{2†} and Stella Mahler²

¹ Institute of Discrete Mathematics and Geometry, TU Wien, Austria
{baaz,mariami}@logic.at

² Institute of Logic and Computation, TU Wien, Austria
{anela,stella}@logic.at

Abstract

First-order logic extended with Henkin quantifiers is not recursively enumerable, even for the simplest Henkin quantifier that is not definable in first-order logic. Despite this, such quantifiers arise naturally in the analysis of partially ordered quantification, motivating partial proof systems suitable for automated reasoning. We present a tableau-based proof system with unification for first-order logic extended by the simplest Henkin quantifier. The system is derived from a second-order function calculus but avoids second-order unification by internalizing dependency structure via side-variable and weak regularity constraints.

1 Introduction

Henkin introduced the general concept of dependent quantifiers as an extension of classical first-order logic [8] (see [9] for an overview). This gives rise to partially ordered quantifiers consisting of m universal and n existential quantifiers, where a function specifies, for each existential quantifier, the universal quantifiers on which it depends (with m, n arbitrary but finite). The simplest Henkin quantifier not definable in ordinary first-order logic is the quantifier Q_H , which binds four variables. A formula using Q_H can be written as

$$\left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) A(x, y, u, v).$$

This is to be read as: “For every x there exists a u , and for every y there exists a v (depending only on y) such that $A(x, y, u, v)$ holds.” Semantically, this formula is equivalent to the second-order formula

$$\exists f \exists g \forall x \forall y A(x, y, f(x), g(y)),$$

where f and g are function variables.

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Systems of partially ordered quantification are intermediate between first-order and second-order logic. As in the case of second-order logic, first-order logic extended by Q_H is incomplete [12]. In proof theory, such incomplete logics are typically treated via partial proof systems, as exemplified by the extensive literature on proof systems for second-order logic. In analytic settings, partial proof systems enable the extraction of implicit computational information from proofs, as in proof mining. However, in contrast to second-order logic, comparatively few results address the proof-theoretic treatment of dependent quantifiers within partial systems. Until recently, the most relevant contribution was due to López-Escobar [10], who developed a system for Q_H in an intuitionistic natural deduction setting. Due to the lack of an elimination rule corresponding to the introduction rule, this system does not admit normalization.

A first step toward an analytic calculus in sequent format for Q_H is to establish an analytic calculus with a suitable partial Henkin semantics. In [2] such a function calculus was introduced: **LF** is obtained by adding to the usual propositional inference rules of **LK** the quantifier inference rules:

- $\forall$ -introduction for second-order function variables

$$\frac{A(t_1^*, \dots, t_n^*), \Gamma \rightarrow \Delta}{\forall f^* A(f^*(t_1^*, \dots, t_n^*)), \Gamma \rightarrow \Delta} \forall_l^n$$

Here t is a term and $t_1^*, \dots, t_n^*$ are terms with free variables¹.

$$\frac{\Gamma \rightarrow \Delta, A(f(t_1^*, \dots, t_n^*))}{\Gamma \rightarrow \Delta, \forall f^* A(f^*(t_1^*, \dots, t_n^*))} \forall_r^n$$

Here f is a free function variable (eigenvariable) of arity n that does not occur in the lower sequent, and $t_1^*, \dots, t_n^*$ are terms with free variables.

- $\exists$ -introduction for second-order function variables

$$\frac{A(f(t_1^*, \dots, t_n^*)), \Gamma \rightarrow \Delta}{\exists f^* A(f^*(t_1^*, \dots, t_n^*)), \Gamma \rightarrow \Delta} \exists_l^n$$

Here f is a free function variable (eigenvariable) of arity n that does not occur in the lower sequent, and $t_1^*, \dots, t_n^*$ are terms with free variables.

$$\frac{\Gamma \rightarrow \Delta, A(t_1^*, \dots, t_n^*)}{\Gamma \rightarrow \Delta, \exists f^* A(f^*(t_1^*, \dots, t_n^*))} \exists_r^n$$

Here t is a term and $t_1^*, \dots, t_n^*$ are terms with free variables.

It was shown in [2] that the calculus **LF** is cut-free complete with respect to term models via the standard Schütte argument and admits effective cut-elimination. While this translation gives a clean semantic representation, proof search in the resulting function calculus requires higher-order unification. Although decidable fragments of higher-order unification exist, they generally involve substantially more complex unification procedures than first-order unification and do not in general provide most general unifiers. We therefore avoid this additional

¹Quantifiers might change the status of a free variable to a bound variable.

proof-search machinery and instead internalize the functional dependencies using first-order unification together with side-variable and weak-regularity constraints.

It was shown in [2] (and later extended to arbitrarily complex Henkin quantifiers in [3]) that analytic sequent calculi in the language of first-order logic can be obtained for Q_H by admitting weakened eigenvariable conditions for these quantifiers. The resulting calculi are globally sound, while allowing locally unsound inferences. These systems provide a proof-theoretic treatment of Henkin quantifiers that avoids explicit second-order reasoning at the level of derivations.

In contrast to sequent calculi, tableau calculi admit a direct operational interpretation and naturally support unification-based proof search. This makes tableaux particularly suitable for automated reasoning with Henkin quantifiers. However, transferring globally sound calculi with locally unsound inferences to tableaux is nontrivial, since tableau closure is branch-local and interacts directly with unification and dependency constraints.

The main contribution of this work is the development of analytic tableau calculi with unification for Henkin quantifiers based on globally sound but locally unsound inference principles. We introduce tableau counterparts of the sequent calculi for Q_H , formulate admissible closure conditions via side-variable dependencies and weak regularity constraints, and establish soundness and completeness relative to the corresponding function calculi.

2 The Analytic Sequent Calculus $\mathbf{LH}^{++}$

To illustrate the need for unsound inferences when dealing with the macros for quantifiers such as Q_H let us first consider the underlying idea of the sequent calculus $\mathbf{LH}^{++}$. A macro for quantifiers is considered as a quantifier in its own right.

Definition 1 (macro for quantifiers). *A macro for quantifiers M is a formula based on quantifiers $Q_i \in \{\forall, \exists\}$, $1 \leq i \leq n$ which is considered as a quantifier in its own right:*

$$M_{x_1, \dots, x_n} A(x_1, \dots, x_n) \equiv Q_1 x_1 \dots Q_n x_n A(x_1, \dots, x_n).$$

For example, we could define a macro $Q_{x,y} A(x, y) \equiv \forall x \exists y A(x, y)$. The idea is to associate with a macro for quantifiers such as $Q_{x,y}$ a left and a right inference rule, and to extend $\mathbf{LK}$ with these rules in such a way that the resulting calculus is analytic.

The calculus for Q_H presented here is a modification of the one in [2]². A first approach to obtain a sequent calculus for first-order logic extended with Q_H would be to add the rules H_r and H_l below to $\mathbf{LK}$, thus obtaining the calculus $\mathbf{LH}$:

$$\frac{\Gamma \rightarrow \Delta, A(a, b, t_1, t_2)}{\Gamma \rightarrow \Delta, \left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) A(x, y, u, v)} H_r$$

a and b are eigenvariables ($a \neq b$) not allowed to occur in the lower sequent and t_1 and t_2 are terms such that a does not occur in t_2 and b does not occur in t_1 . It should be noted that such a rule was already used by Lopez-Escobar in [10]. The left inference (not defined in [10]) is

$$\frac{A(t_1, t_2, a, b), \Gamma \rightarrow \Delta}{\left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) A(x, y, u, v), \Gamma \rightarrow \Delta} H_l$$

²During the development of the tableaux calculi presented in this work, it became apparent that the conditions for unsound inferences, in particular the condition for very weak regularity, must be formulated more precisely than in [2].

where a and b are eigenvariables ($a \neq b$) not allowed to occur in the lower sequent and t_1, t_2 are terms such that b does not occur in t_2 and neither a nor b occurs in t_1 . The rules are obtained from an analysis of the corresponding sequences of inferences in **LF**. In particular, the rule H_r originates from

$$\frac{\frac{\frac{\frac{\Gamma \rightarrow \Delta, A(a, b, t_1(a), t_2(b))}{\Gamma \rightarrow \Delta, \forall y A(a, y, t_1(a), t_2(y))} \forall_r}{\Gamma \rightarrow \Delta, \forall x \forall y A(x, y, t_1(x), t_2(y))} \forall_r}{\Gamma \rightarrow \Delta, \exists g \forall x \forall y A(x, y, t_1(x), g(y))} \exists_r}{\Gamma \rightarrow \Delta, \exists f \exists g \forall x \forall y A(x, y, f(x), g(y))} \exists_r$$

and H_l from

$$\frac{\frac{\frac{\frac{A(t_1, t_2, a(t_1), b(t_2)), \Gamma \rightarrow \Delta}{\forall y A(t_1, y, a(t_1), b(y)), \Gamma \rightarrow \Delta} \forall_l}{\forall x \forall y A(x, y, a(x), b(y)), \Gamma \rightarrow \Delta} \forall_l}{\exists g \forall x \forall y A(x, y, a(x), g(y)), \Gamma \rightarrow \Delta} \exists_l}{\exists f \exists g \forall x \forall y A(x, y, f(x), g(y)), \Gamma \rightarrow \Delta} \exists_l$$

where a, b are eigenvariables. Clearly, $a(t_1)$ cannot occur in t_1 and $b(t_2)$ cannot occur in t_2 . Moreover, $b(t_2)$ does not occur in t_1 .

LH admits cut-elimination, however, it is shown to be incomplete even for trivial statements [3]. The reason is that even simple sequents such as

$$\left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) A(x, y, u, v) \rightarrow \left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) (A(x, y, u, v) \vee C)$$

are not provable in **LH** (in fact, not even for the simpler recursively enumerable Henkin quantifier $(\forall x \exists u)$). If the above sequent was provable, then, by the cut-elimination theorem, it would be provable without cuts. A cut-free derivation after deletion of weakening and contractions would have the form

$$\frac{A(a, b, c, d) \rightarrow A(a, b, c, d)}{A(a, b, c, d) \rightarrow A(a, b, c, d) \vee C} \\ \vdots$$

Due to the mixture of strong and weak positions in Q_H , none of the quantifier inferences for Q_H can be applied. To overcome the problem with incompleteness, **LH**⁺⁺ was introduced in [3] as an extension of the calculus **LK**⁺⁺. Before we describe **LH**⁺⁺, let us first have a look at the underlying idea of unsound calculi.

2.1 The Globally Sound Calculus **LK**⁺⁺

The inherent incompleteness for trivial statements involving Q_H is based on the fact that Q_H represents a quantifier inference macro that combines quantifiers in a strong³ and a weak position. This phenomenon occurs already at the level of usual first-order logic when quantifiers defined by macros of quantifiers such as $\forall x \exists y$ are considered [4]. The solution is to consider

³A strong quantifier in a formula A is a universal quantifier which occurs in a positive position in A , or an existential quantifier which occurs in a negative position in A . All other quantifiers are called weak.

sequent calculi with concepts of proof which are globally but not locally sound. This means that every derived statement is true, although not every sub-derivation is meaningful. The following definitions are from [1].

Definition 2 (side-variable relation for **LK**). *Let φ be an **LK**-derivation. We say b is a side-variable of a in φ (denoted by $a < b$) if φ contains a strong quantifier inference of the form*

$$\frac{\Gamma \rightarrow \Delta, A(a, b, \bar{c})}{\Gamma \rightarrow \Delta, \forall x A(x, b, \bar{c})} \forall_r$$

or

$$\frac{A(a, b, \bar{c}), \Gamma \rightarrow \Delta}{\exists x A(x, b, \bar{c}), \Gamma \rightarrow \Delta} \exists_l$$

Definition 3 (**LK**⁺⁺-suitable quantifier inferences). *We say a quantifier inference is suitable for a proof φ if either it is a weak quantifier inference, or the following three conditions are satisfied:*

- (substitutability) *the eigenvariable does not appear in the conclusion of φ .*
- (side-variable condition) *the relation $<$ is acyclic.*
- (very weak regularity) *the eigenvariable of an inference with main formula A is different to the eigenvariable of an inference with main formula A' whenever $A \neq A'$.*

We obtain **LK**⁺⁺ from **LK** by replacing the usual eigenvariable conditions by **LK**⁺⁺-suitable ones.

Acyclic dependency conditions are a standard proof-theoretic device and occur, for example, in the correctness criteria for Girard's proof nets [7] and in dependency conditions associated with Herbrand-style representations [11]. Our use of acyclicity is related, the relation records dependencies introduced by quantifier inferences and rules out cyclic dependencies that would prevent a sound global interpretation of the derivation. In our work, however, the dependency relation arises specifically from the functional dependencies encoded by Henkin quantifiers and is combined with the very weak regularity condition.

It was shown in [1] (Theorem 2.8.) that **LK**⁺⁺ is globally sound.

Theorem 1. *If a sequent is **LK**⁺⁺-derivable, then it is already **LK**-derivable.*

The proof proceeds by replacing unsound quantifier inferences by implication inferences and introduction of cuts.

The main property of such calculi is that they preserve the grouping of inferences in the cut-free format that occur together in the format with cuts. This format induces quantifier shifts, and such calculi are not sound unless all classical quantifier shifts are available.

Example 1. *The following **LK**⁺⁺-derivation is locally unsound but globally sound:*

$$\frac{\frac{\frac{A(a) \rightarrow A(a)}{A(a) \rightarrow \forall y A(y)} \forall_r}{\rightarrow A(a) \supset \forall y A(y)} \supset_r}{\rightarrow \exists x (A(x) \supset \forall y A(y))} \exists_r$$

As a is the only eigenvariable, the side-variable relation is empty.

2.2 The Globally Sound Calculus $\mathbf{LH}^{++}$

As the quantifier inference rules of $\mathbf{LH}$ are based on the quantifier inference rules of $\mathbf{LF}$, $\mathbf{LH}^{++}$ is based on the corresponding function calculus $\mathbf{LF}^{++}$. Moreover, $\mathbf{LF}^{++}$ is obtained from $\mathbf{LF}$ in the same way as $\mathbf{LK}^{++}$ is obtained from $\mathbf{LK}$, by replacing the quantifier inferences by suitable ones as in Definition 3.

Theorem 2. *Every $\mathbf{LF}^{++}$ -derivable sequent is already $\mathbf{LF}$ -derivable.*

Proof. By the same construction as in the proof of Theorem 1. $\square$

Note that derivations in $\mathbf{LK}$ are easily convertible into derivations in $\mathbf{LK}^{++}$, since under the usual eigenvariable conditions the side-variable conditions are acyclic by construction, and even strong regularity holds. This also holds for derivations in $\mathbf{LF}$ that are easily convertible into derivations in $\mathbf{LF}^{++}$.

Definition 4 (side-variable relation for $\mathbf{LH}$). *The side-variable relations are obtained from the inference of the corresponding macros in $\mathbf{LF}$. For*

$$\frac{\Gamma \rightarrow \Delta, A(a, b, t_1(a, \bar{c}), t_2(b, \bar{c}), \bar{c})}{\Gamma \rightarrow \Delta, \left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) A(x, y, u, v, \bar{c})} H_r$$

a does not occur in t_2 and b does not occur in t_1 . We order the variables

$$b < a$$

and for all other free variables $c \in \bar{c}$,

$$\{a, b\} < c,$$

where $\Gamma < \Delta$ means that $\forall c \in \Gamma$ and $\forall d \in \Delta$, $c < d$. Similarly, consider

$$\frac{A(t_1, t_2, a, b, \bar{c}), \Gamma \rightarrow \Delta}{\left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) A(x, y, u, v, \bar{c}), \Gamma \rightarrow \Delta} H_l$$

a does not occur in t_2 and a and b do not occur in t_1 or b not in t_1 and a and b not in t_2 . We order the variables

$$b < a$$

and for all other free variables $c \in \bar{c}$,

$$\{a, b\} < c.$$

The definition below is due to the macros for Henkin quantifiers in $\mathbf{LF}^{++}$.

Definition 5 (very weak regularity for $\mathbf{LH}^{++}$). *Consider a derivation with two occurrences of a Henkin rule instance of the same polarity. Let a, b be the eigenvariables such that b corresponds to the innermost strong quantifier and a to the outermost strong quantifier in the macro Q_H . This yields a matrix*

$$A(a, b, x, y)$$

in the premise for the right rule H_r , and a matrix

$$A(x, y, a, b)$$

for the left rule H_l . For inferences with an intersection of eigenvariables very weak regularity holds if the following conditions hold:

- If the innermost eigenvariable b occurs as innermost eigenvariable in another inference, then the corresponding outermost eigenvariable a occurs as the outermost eigenvariable in that inference too, i.e. repeating b forces repeating a .
- If the outermost eigenvariables are the same for several inferences, the premise formula is the same in these inferences modulo the innermost eigenvariable (as innermost eigenvariables may differ).

Definition 6 ($\mathbf{LH}^{++}$ -suitable quantifier inferences). A quantifier inference is $\mathbf{LH}^{++}$ -suitable for a proof φ if the following conditions are satisfied:

- (substitutability) the eigenvariable does not appear in the conclusion of φ .
- (side-variable condition) the relation $<$ is acyclic.
- (very weak regularity).

The sequent calculus $\mathbf{LH}^{++}$ is $\mathbf{LH}$, except that we replace quantifier inferences with $\mathbf{LH}^{++}$ -suitable quantifier inferences. $\mathbf{LH}^{++}$ is sound and complete relative to the second-order calculus for functions $\mathbf{LF}$, and admits cut-elimination [3].

Theorem 3 (Cf. Corollary 4. and Corollary 5. in [3]). A sequent with macros for Henkin quantifiers in their second-order representation using functions is derivable in $\mathbf{LF}$ iff its translation into a corresponding sequent with Henkin quantifiers Q_H is cut-free derivable in $\mathbf{LH}^{++}$.

Proof. ($\Rightarrow$) $\mathbf{LF}$ admits cut-elimination and in cut-free proofs in $\mathbf{LF}$ the inferences can be shifted to obtain macros for Henkin quantifiers with proofs in $\mathbf{LF}^{++}$. These proofs can be immediately transformed into cut-free proofs with Q_H in $\mathbf{LH}^{++}$.

($\Leftarrow$) Given a cut-free proof in $\mathbf{LH}^{++}$ replace the eigenvariable in H_l inferences top-down such that the function variable for the inference is distinct from other function variables in the terms for different eigenvariables. Both, the non-looping of the side-variable conditions and the very weak regularity are preserved. We obtain a derivation in $\mathbf{LF}^{++}$, which can be transformed into a derivation in $\mathbf{LF}$ by additional cuts as in Theorem 2. $\square$

Example 2. Consider the cut-free derivation $\varphi =$

$$\begin{array}{c}
 \frac{A(a, b, c, d) \rightarrow A(a, b, c, d)}{A(a, b, c, d) \rightarrow A(a, b, c, d), C} w_r \\
 \frac{A(a, b, c, d) \rightarrow A(a, b, c, d), C}{A(a, b, c, d) \rightarrow A(a, b, c, d) \vee C} \vee_r \\
 \frac{A(a, b, c, d) \rightarrow A(a, b, c, d) \vee C}{A(a, b, c, d) \rightarrow \left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) (A(x, y, u, v) \vee C)} H_r \\
 \frac{A(a, b, c, d) \rightarrow \left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) (A(x, y, u, v) \vee C)}{\left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) A(x, y, u, v) \rightarrow \left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) (A(x, y, u, v) \vee C)} H_l
 \end{array}$$

The eigenvariables a, b, c, d do not occur in the end-sequent, very weak regularity is preserved, and the side-variable relation does not loop. Indeed, we obtain the side-variable relations

$$b < a, \{a, b\} < \{c, d\}$$

from the right inference and from the left inference we obtain $d < c$.

3 Analytic Tableaux with Unification

The analytic tableau method for first-order logic is a well-established proof procedure based on the systematic decomposition of formulas [5]. A tableau is a finitely branching tree whose nodes are labeled with formulas. Starting from an initial formula, typically in negated form, the tree is expanded by applying decomposition rules that reduce complex formulas to their sub-formulas along individual branches.

Analytic tableaux admit a close correspondence with sequent calculi. In particular, a tableau may be viewed as a decomposition of a sequent in which truth and falsity information is represented explicitly. To make this connection precise, we employ signed formulas.

Definition 7 (signed formulas). *A signed formula is an expression of the form*

$$\mathbf{T}A \quad \text{or} \quad \mathbf{F}A,$$

where A is a first-order formula. The sign $\mathbf{T}$ indicates that A is asserted to be true, whereas $\mathbf{F}$ indicates that A is asserted to be false.

In this interpretation, formulas occurring on the left-hand side of a sequent are assigned the sign $\mathbf{T}$, whereas formulas on the right-hand side are assigned the sign $\mathbf{F}$. Thus a sequent

$$\Gamma \rightarrow \Delta$$

corresponds to the set of signed formulas

$$\{\mathbf{T}A : A \in \Gamma\} \cup \{\mathbf{F}B : B \in \Delta\}.$$

Under this correspondence, tableau expansion rules can be understood as backward applications of sequent inference rules. As a consequence, many structural properties of sequent calculi have natural tableau counterparts. However, when passing from globally sound but potentially locally unsound sequent calculi for Q_H to tableau systems, admissibility conditions must be reformulated in terms of branch closure and unification. This translation is therefore nontrivial.

The standard first-order tableau calculus (cf. [6]) employs the usual propositional decomposition rules together with eigenconstants for strong quantifier instantiation and function symbols for Skolem terms. The systems developed in this work retain the propositional component of the standard tableau calculus while introducing modified rules for quantifiers.

Definition 8 (propositional tableau). *The signed propositional tableau is a tree whose nodes are signed formulas of the form $\mathbf{T}A$ or $\mathbf{F}A$ and is constructed by repeatedly applying the following decomposition rules:*

$\frac{\mathbf{T}(A \wedge B)}{\mathbf{T}(A) \quad \mathbf{T}(B)}$	$\frac{\mathbf{F}(A \wedge B)}{\mathbf{F}(A) \mid \mathbf{F}(B)}$
$\frac{\mathbf{T}(A \vee B)}{\mathbf{T}(A) \mid \mathbf{T}(B)}$	$\frac{\mathbf{F}(A \vee B)}{\mathbf{F}(A) \quad \mathbf{F}(B)}$
$\frac{\mathbf{T}(A \supset B)}{\mathbf{F}(A) \mid \mathbf{T}(B)}$	$\frac{\mathbf{F}(A \supset B)}{\mathbf{T}(A) \quad \mathbf{F}(B)}$

$$\frac{\mathbf{T}(\neg A)}{\mathbf{F}(A)} \qquad \frac{\mathbf{F}(\neg A)}{\mathbf{T}(A)}$$

A branch is a maximal path through the tableau tree.

Definition 9 (closure of propositional tableau). A branch is closed if it contains both

$$\mathbf{T}A \quad \text{and} \quad \mathbf{F}A$$

for some formula A . A tableau is closed if every branch is closed, in which case it constitutes a refutation of the initial formula.

Since analytic tableaux are cut-free by construction, they correspond closely to cut-free sequent derivations. Consequently, proof-theoretic phenomena that affect cut-free proofs are reflected in tableau systems as well. For example, there is a non-elementary speed-up between tableaux corresponding to $\mathbf{LK}$ and $\mathbf{LK}^{++}$ [1]. Similar phenomena arise in the presence of Henkin quantifiers.

We first introduce the unification-based tableau calculus $\mathbf{Tab}_U(\mathbf{LK}^{++})$, which accommodates globally sound but potentially locally unsound inference rules, and subsequently extend it to obtain $\mathbf{Tab}_U(\mathbf{LH}^{++})$, a tableau calculus for Henkin quantifiers.

3.1 The Tableau Calculus with Unification $\mathbf{Tab}_U(\mathbf{LK}^{++})$

The essential distinction between ordinary first-order tableaux with unification and the tableau calculus $\mathbf{Tab}_U(\mathbf{LK}^{++})$ concerns the treatment of quantifiers. As in free-variable tableau calculi, weak quantifier rules introduce fresh term variables, whereas strong quantifier rules introduce eigenconstants. However, instead of the usual unification conditions, $\mathbf{Tab}_U(\mathbf{LK}^{++})$ employs a weaker admissibility requirement, called very weak regularity.

To formalize these conditions, we associate with each tableau a constraint set, analogous to the side-variable conditions used in the sequent calculi introduced in the previous sections, that records the dependencies generated by applications of strong quantifier rules. One of the conditions for the tableau closure to be permitted is that these dependency constraints form an acyclic relation.

Definition 10 (constraint set). Associated with every tableau is a constraint set

$$C \subseteq \mathcal{E} \times (\mathcal{E} \cup \mathcal{V}),$$

where $\mathcal{E}$ denotes the set of eigenconstants and $\mathcal{V}$ the set of term variables occurring in the tableau. We write

$$c < x$$

whenever $(c, x) \in C$. Thus, (C) induces a binary relation $(<)$ on $(\mathcal{E} \times (\mathcal{E} \cup \mathcal{V}))$.

Initially, $C = \emptyset$. Each application of a strong quantifier rule extends C by adding the corresponding constraints associated with this rule. Note that for an eigenconstant $c \in \mathcal{E}$ and a term variable $X \in \mathcal{V}$, we write

$$c < X$$

to mean that $c < d$ for every eigenconstant $d \in X$, where $d \in X$ indicates that d occurs in the term represented by X .

Definition 11 ($\mathbf{Tab}_U(\mathbf{LK}^{++})$). The tableau calculus with unification $\mathbf{Tab}_U(\mathbf{LK}^{++})$ is obtained from propositional tableau by the addition of the following decomposition rules:

Decomposition for weak quantifier rules

$$\frac{\mathbf{T}(\forall x. A(x))}{\mathbf{T}(A[Z/x])} \qquad \frac{\mathbf{F}(\exists x. A(x))}{\mathbf{F}(A[Z/x])}$$

where Z is a fresh term variable.

Decomposition for strong quantifier rules

$$\frac{\mathbf{F}(\forall x. A(x, \bar{c}, \bar{Z}))}{\mathbf{F}(A([a/x], \bar{c}, \bar{Z}))} \qquad \frac{\mathbf{T}(\exists x. A(x, \bar{c}, \bar{Z}))}{\mathbf{T}(A([a/x], \bar{c}, \bar{Z}))}$$

where a is a fresh eigenconstant with respect to the entire tableau. Each $c \in \bar{c}$ is called a side-constant of a , written as

$$a < c.$$

Moreover, every term variable $Z \in \bar{Z}$ is a side-variable of a , i.e.

$$a < Z.$$

The dependencies $a < c$ and $a < Z$ are added to the constraint set C .

Since tableau closure in $\mathbf{Tab}_U(\mathbf{LK}^{++})$ is based on unification, we first define the notion of an admissible substitution. A closed tableau must admit a substitution that satisfies both the dependency relations encoded in C and the very weak regularity condition.

Definition 12 (admissible substitution for $\mathbf{Tab}_U(\mathbf{LK}^{++})$). *Let C be the constraint set associated with a tableau. A substitution σ is called admissible if the following conditions are satisfied:*

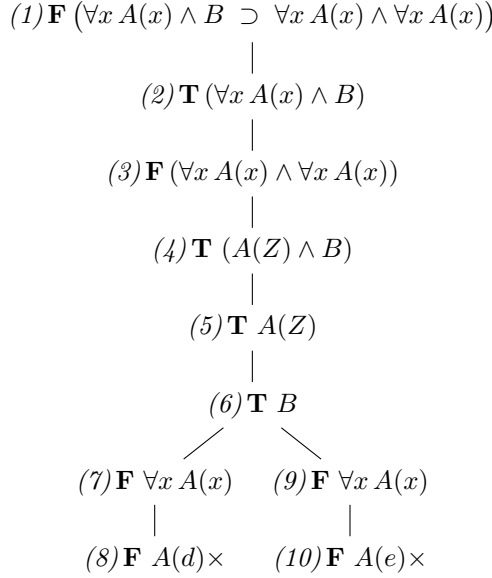
1. *Term variables may be substituted with terms in the usual way.*
2. *Eigenconstants may only be substituted with eigenconstants.*
3. *The dependency relation induced by C remains acyclic after applying σ .*
4. *(very weak regularity) If two inferences have the same eigenconstant after applying σ , then their main formulas coincide. Equivalently, inferences with distinct main formulas must have distinct eigenconstants.*

Definition 13 (closure of $\mathbf{Tab}_U(\mathbf{LK}^{++})$). *A branch is closed if it contains literals*

$$\mathbf{TA} \quad \text{and} \quad \mathbf{FB}$$

such that $A\sigma = B\sigma$ for some admissible substitution σ . A tableau is closed if all of its branches are closed under a common admissible substitution.

Example 3. *The following tableau is closed in $\mathbf{Tab}_U(\mathbf{LK}^{++})$.*



The admissible substitution is:

$$\sigma = \{Z \leftarrow e, d \leftarrow e\}.$$

Indeed,

$$A(Z)\sigma = A(e), \quad A(d)\sigma = A(e), \quad A(e)\sigma = A(e).$$

The constraint set C remains empty in this example and is therefore trivially acyclic. Very weak regularity is preserved under σ , as for the eigenconstants d and e we have $d\sigma = e\sigma$, but their main formulas $\mathbf{F} \forall x A(x)$ are identical.

Theorem 4. $\mathbf{Tab}_U(\mathbf{LK}^{++})$ is sound and complete with respect to $\mathbf{LK}^{++}$.

Proof. The result follows from the correspondence between cut-free derivations in $\mathbf{LK}^{++}$ and closed tableaux in $\mathbf{Tab}_U(\mathbf{LK}^{++})$, together with the soundness and completeness results established in [1]. $\square$

The significance of $\mathbf{Tab}_U(\mathbf{LK}^{++})$ is that it supports analytic proof search based entirely on first-order unification. This feature becomes particularly important in the presence of Henkin quantifiers. Although Henkin quantifiers can be encoded in second-order logic by means of function quantification, proof search in such formalisms generally requires higher-order unification, which is undecidable in general. The tableau calculus $\mathbf{Tab}_U(\mathbf{LH}^{++})$, introduced in the next section, avoids this problem by extending first-order unification techniques directly to Henkin quantifiers.

4 The Tableau Calculus with Unification $\mathbf{Tab}_U(\mathbf{LH}^{++})$

The tableau calculus $\mathbf{Tab}_U(\mathbf{LH}^{++})$ extends the unification-based first-order tableau calculus $\mathbf{Tab}_U(\mathbf{LK}^{++})$ by incorporating analytic decomposition rules for Henkin quantifiers. As in $\mathbf{Tab}_U(\mathbf{LK}^{++})$, quantifier inferences are allowed to be globally sound while potentially locally unsound. Correctness is ensured by means of dependency constraints recorded in the constraint set together with a suitable regularity condition on eigenconstants.

The use of unification remains central to the calculus. Without unification, proof search would require explicit witness instantiations during tableau construction. Instead, the calculus $\mathbf{Tab}_U(\mathbf{LH}^{++})$ introduces term variables and postpones instantiation choices until branch closure. As in free-variable tableau calculi, this allows proof search to proceed analytically while significantly reducing the amount of nondeterminism.

Definition 14 ($\mathbf{Tab}_U(\mathbf{LH}^{++})$). *The tableau calculus with unification $\mathbf{Tab}_U(\mathbf{LH}^{++})$ is obtained from $\mathbf{Tab}_U(\mathbf{LK}^{++})$ by the addition of the following decomposition rules for Henkin quantifiers:*

Negative Henkin decomposition

$$\frac{\mathbf{F}\left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array}\right) A(x, y, u, v, \bar{c}, \bar{V})}{\mathbf{F}A(a, b, Z_1, Z_2, \bar{c}, \bar{V})}$$

where a, b are fresh eigenconstants ($a \neq b$), each $c \in \bar{c}$ is a side-constant, Z_1, Z_2 are fresh term variables, and each $V \in \bar{V}$ is a side-variable, such that

$$a \notin Z_2, \quad b \notin Z_1^4,$$

and the following constraints are added to C :

1. $b < a$,
2. $\{a, b\} < c$,
3. $\{a, b\} < V$,
4. $\{a, b\} <_a Z_1, \{a, b\} <_b Z_2$,

where $\Gamma <_a Z$ means that for every eigenconstant $d \in Z$ with $d \neq a$ it holds that $\Gamma < d$.

Positive Henkin decomposition

$$\frac{\mathbf{T}\left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array}\right) A(x, y, u, v, \bar{c}, \bar{V})}{\mathbf{T}A(Z_1, Z_2, a, b, \bar{c}, \bar{V})}$$

where a, b are fresh eigenconstants ($a \neq b$), Z_1, Z_2 are fresh term variables, each $c \in \bar{c}$ is a side-eigenconstant, and each $V \in \bar{V}$ is a side-variable, such that

$$a \notin Z_2, \quad a \notin Z_1, \quad b \notin Z_1, \quad \text{or} \quad a \notin Z_2, \quad b \notin Z_1, \quad b \notin Z_2^5,$$

and the following constraints are added to C :

1. $b < a$,

⁴ a cannot occur in Z_2 and b cannot occur in Z_1 for the same reason as in the definition of side-variable relations for $\mathbf{LH}$ in Definition 4, the same holds for items 1. and 2. of the constraints below. Case 3. and 4. are specific for the tableau calculus.

⁵ a does not occur in Z_2 and a and b do not occur in Z_1 , or b does not occur in Z_1 and a and b do not occur in Z_2 , for the same reason as in the definition of side-variable relations for $\mathbf{LH}$ in Definition 4. The same holds for items 1. and 2. of the constraints below. Item 3. is specific for the tableau calculus.

2. $\{a, b\} < c$,
3. $\{a, b\} < V$.

In addition to the acyclicity condition on the constraint set, closure in $\mathbf{Tab}_U(\mathbf{LH}^{++})$ is subject to a strengthened form of very weak regularity that accounts for the nested eigenconstants introduced by Henkin decomposition rules.

Definition 15 (very weak regularity for $\mathbf{Tab}_U(\mathbf{LH}^{++})$). *Consider a tableau derivation in $\mathbf{Tab}_U(\mathbf{LH}^{++})$ containing two applications of Henkin decomposition rules of the same polarity. Suppose that these rule instances introduce eigenconstants a and b , where a corresponds to the outermost strong quantifier and b to the innermost strong quantifier of the Henkin quantifier macro. The corresponding premises are of the form*

- $A(a, b, Z_1, Z_2, \bar{c}, \bar{V})$ (negative decomposition),
- $A(Z_1, Z_2, a, b, \bar{c}, \bar{V})$ (positive decomposition).

For inferences with an intersection of eigenconstants, very weak regularity holds if the following conditions are satisfied:

- If an eigenconstant b occurs as innermost eigenconstant in two rule instances, then the corresponding outer eigenconstant a occurs as outer eigenconstant in both instances.
- If two rule instances have the same outer eigenconstant a , then their premises coincide modulo renaming of the innermost eigenconstant b .

To define tableau closure, we again introduce the notion of admissible substitution.

Definition 16 (admissible substitution for $\mathbf{Tab}_U(\mathbf{LH}^{++})$). *Let C be a constraint set associated with a tableau. A substitution σ is called admissible if the following conditions are satisfied:*

1. Term variables may be substituted with terms in the usual way.
2. Eigenconstants may only be substituted with eigenconstants.
3. The constraint set relations induced by C remain acyclic after applying σ .
4. The very weak regularity for $\mathbf{Tab}_U(\mathbf{LH}^{++})$ is preserved.

Example 4. *Let us illustrate an admissible substitution on a derivation of the sequent*

$$\left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) A(x, y, u, v) \rightarrow \left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) (A(x, y, u, v) \vee C)$$

from Example 2:

$$\begin{array}{c}
(1) \mathbf{F} \left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) A(x, y, u, v) \supset \left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) (A(x, y, u, v) \vee C) \\
\downarrow \\
(2) \mathbf{T} \left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) A(x, y, u, v) \\
\downarrow \\
(3) \mathbf{F} \left(\begin{array}{cc} \forall x & \exists u \\ \forall y & \exists v \end{array} \right) (A(x, y, u, v) \vee C) \\
\downarrow \\
(4) \mathbf{F} A(a, b, Z_1, Z_2) \vee C \\
\downarrow \\
(5) \mathbf{F} A(a, b, Z_1, Z_2) \\
\downarrow \\
(6) \mathbf{F} C \\
\downarrow \\
(7) \mathbf{T} A(X_1, X_2, c, d) \times
\end{array}$$

The constraint set generated during the tableau derivation is given below, grouped according to the corresponding inference steps:

(4) $b < a$, $\{a, b\} <_a Z_1$, $\{a, b\} <_b Z_2$ (note that C does not contain any eigenconstants or term variables that would have to be considered as side-eigenconstants or side-variables here).

(7) $d < c$.

Additionally, rule (4) requires that

$$a \notin Z_2, \quad b \notin Z_1,$$

and from rule (7) it follows that

$$c \notin X_2, \quad c \notin X_1, \quad d \notin X_1.$$

An admissible substitution for this example is $\{X_1 \leftarrow a, X_2 \leftarrow b, Z_1 \leftarrow c, Z_2 \leftarrow d\}$. $\times$ denotes that the branch is closed, to be defined below in Definition 17.

Definition 17 (closure of $\mathbf{Tab}_U(\mathbf{LH}^{++})$). A branch is closed if it contains literals

$$\mathbf{T}A \quad \text{and} \quad \mathbf{F}B$$

from rule (5) it follows that

$$c \notin X_2, c \notin X_1, d \notin X_1,$$

and from rule (6) it follows that

$$e \notin Y_2, e \notin Y_1, f \notin Y_1.$$

An admissible substitution for this example is

$$\{X_1 \leftarrow a, X_2 \leftarrow b, Z_1 \leftarrow c, Z_2 \leftarrow d\}.$$

5 Conclusion

We introduced an analytic tableau calculus for first-order logic extended with the Henkin quantifier Q_H . The calculi developed in this work are based on globally sound but locally unsound inference principles originating from corresponding sequent and function calculi.

The tableau systems developed in this work provide a proof-search oriented counterpart to the previously studied sequent calculi. In particular, the calculi support first-order style unification together with dependency constraints that can be checked operationally during tableau construction. This yields a framework for automated reasoning with Henkin quantifiers that remains within a first-order setting despite the underlying second-order semantics.

The methods presented here extend naturally to more general classes of Henkin quantifiers and to stronger function calculi. More generally, the approach suggests a systematic method for constructing analytic first-order tableau systems for second-order definable quantifiers while retaining first-order proof-search mechanisms.

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