Representation of the Medial-Like Algebras

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Abstract

In this paper we characterize the regular medial algebras, the paramedial n-ary groupoids with a regular element, the paramedial algebras with a regular element and the regular paramedial algebras. Also, we characterize paramedial, co-medial and co-paramedial pairs of quasigroup operations and paramedial, co-medial and co-paramedial algebras with the quasigroup operations.

1 Introduction

An algebra, \( A = (A,F) \), (without nullary operations) is called medial (entropic, abelian) if it satisfies the identity of mediality:

\[
g(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1m}, \ldots, x_{nm})) = \]
\[
f(g(x_{11}, \ldots, x_{1m}), \ldots, g(x_{n1}, \ldots, x_{nm})),
\]

for every \( n \)-ary \( f \in F \) and \( m \)-ary \( g \in F \) [5]. The \( n \)-ary operation, \( f \), is called idempotent if \( f(x, \ldots, x) = x \), for every \( x \in A \). The algebra \( A = (A,F) \) is called idempotent, if every operation \( f \in F \) is idempotent. An idempotent medial algebra is a mode [10].

Let \( g \) and \( f \) be \( m \)-ary and \( n \)-ary operations on the set, \( A \). We say that the pair of operations, \( (f,g) \), is medial (entropic), if the identity [1] holds in the algebra, \( A = (A,f,g) \).

We say that the pair of operations, \( (f,g) \), is paramedial (or paraentropic), if the following identity holds in the algebra, \( A = (A,f,g) \):

\[
g(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1m}, \ldots, x_{nm})) = \]
\[
f(g(x_{nm}, \ldots, x_{n1}), \ldots, g(x_{1m}, \ldots, x_{11})).
\]

An algebra, \( A = (A,F) \), (without nullary operations) is called paramedial if every pair of operations, \( f,g \in F \), (not necessarily distinct) is paramedial.

Paramedial groupoids and paramedial quasigroups were studied in [1] [9] [11].

Let \( g \) and \( f \) be \( n \)-ary operations on the set, \( A \). We say that the pair of \( n \)-ary operations, \( (f,g) \), is co-medial, if the following identity holds in the algebra, \( A = (A,f,g) \):

\[
g(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1n}, \ldots, x_{nn})) =
\]
\[
g(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{n1}, \ldots, x_{nn})).
\]

The pair of \( n \)-ary operations, \( (f,g) \), is co-paramedial, if the following identity holds in the algebra, \( A = (A,f,g) \):

\[
g(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1n}, \ldots, x_{nn})) =
\]
\[
g(f(x_{n1}, \ldots, x_{n1}), \ldots, f(x_{1n}, \ldots, x_{11})).
\]

An algebra, \( A = (A,F) \), is called a co-medial (co-paramedial) algebra, if every pair of the operations, \( f,g \in F \), with the same arity is co-medial (co-paramedial).

In other words, the algebra, \( A \), is medial (paramedial, co-medial, or co-paramedial) if it satisfies the hyperidentity of mediability (paramediality, co-mediality, or co-paramediality) [7] [8] [6].
2 Main Results

Let \( A = (A,F) \) be an algebra and \( f \in F \). We say that the element \( e \), is the unit for the operation \( f \in F \), if: \( f(x,e,\ldots,e) \approx f(e,x,e,\ldots,e) \approx \ldots \approx f(e,\ldots,e,x) \approx x \), for every \( x \in A \). The element \( e \), is a unit for the algebra \((A,F)\), if it is a unit for every operation, \( f \in F \). The element \( e \), is idempotent for the operation \( f \), if: \( f(e,\ldots,e) = e \). We say that the element \( e \), is idempotent for the algebra \((A,F)\), if it is an idempotent for every operation \( f \in F \).

**Definition 2.1.** Let \((f,g)\) be a pair of \(m\)-ary and \(n\)-ary operations of the algebra, \((A,F)\).

For any element \(e\) of \(A\), let \(\alpha_1, \ldots, \alpha_m\) be mappings of \(A\) into \(A\) defined by

\[ \alpha_i : x \mapsto f(e,\ldots,e,x,e,\ldots,e), \]

with \(x\) at the \(i\)-th place. We call \(\alpha_i\) the \(i\)-th translation by \(e\) with respect to \(f\). An element \(e\) is called \(i\)-regular with respect to \(f\) if \(\alpha_i\) is a bijection. An element \(e\), is called \(i\)-regular for the pair operation, \((f,g)\), if it is an \(i\)-regular with respect to the both operations \(f\) and \(g\). The element \(e\), is called \(i\)-regular for the algebra \((A,F)\), if it is an \(i\)-regular element for every operation \(f \in F\).

The element \(e\) is called regular with respect to the \(n\)-ary operation \(f \in F\), if \(e\) is an \(i\)-regular element with respect to \(f\) for every \(1 \leq i \leq n\). The element \(e\) is a regular element of the algebra \((A,F)\), if \(e\) is a regular element with respect to the every operation \(f \in F\).

**Theorem 2.2.** Let \((A,F)\) be a medial algebra with the idempotent element \(e\) which is \(i\)- and \(j\)-regular element of \((A,F)\) for fixed \(i\) and \(j\) \((i \neq j)\), then there exists a commutative semigroup \((A,+)\) with the unit element \(e\), such that every operation \(f \in F\) has the following linear representation:

\[ f(x_1,\ldots,x_m) = \gamma_1 x_1 + \cdots + \gamma_m x_m, \]

where \(\gamma_1, \ldots, \gamma_m\), are pairwise commuting endomorphisms of \((A,+)\), \(m \geq 2\). Furthermore, \(\gamma_i, \gamma_j\) are automorphisms.

**Definition 2.3.** Let \(f\) be an \(m\)-ary operation and \(J\) be a non-empty subset of \(\{1,2,\ldots,m\}\), we will say that the element \(e\) is \(J\)-regular with respect to the operation \(f\), if \(e\) is a \(j\)-regular element with respect to \(f\), for all \(j \in J\). The element \(e\) is \(J\)-regular element of the algebra \((A,F)\), if \(e\) is a \(J\)-regular element with respect to every \(f \in F\), for all \(j \in J\), where \(m = \min\{g \mid g \in F\}\), and \(m \geq 2\).

**Definition 2.4.** Let \(f, g \in F\) be \(m\)-ary and \(n\)-ary operations \((m \leq n)\), \(J \subseteq \{1,2,\ldots,m\}\) (where \(J\) contains at least two elements) and \(a_1, a_1, a_{i-1}, a_{i+1}, \ldots, a_m, a_n\) are \(J\)-regular elements of the algebra \((A,f,g)\). The pair operation \((f,g)\) is \((i,J)\)-regular pair operation (where \(i \in J\)), if for every \(x \in A\) we have the following equation:

\[ f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m) = g(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) \]

The pair operation \((f,g)\) is a \(J\)-regular pair operation if \((f,g)\) is \((i,J)\)-regular for every \(i \in J\).

The pair operation \((f,g)\) is a regular pair operation if \((f,g)\) is \(J\)-regular pair operation for some \(J \subseteq \{1,2,\ldots,m\}\) (where \(J\) contains at least two elements).

An algebra \((A,F)\) is called a regular algebra if every pair operation of \((A,F)\) be a regular pair operation.

**Theorem 2.5.** Let \((A,f,g)\) be a regular medial algebra with \(m\)-ary operation \(f\) and \(n\)-ary operation \(g\) \((m \leq n)\), then there is a commutative semigroup with an unit element \((A,+)\), such that

\[ f(x_1,\ldots,x_m) = \gamma_1 x_1 + \cdots + \gamma_m x_m + d_1, \]
\[ g(x_1,\ldots,x_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n + d_2, \]
where, \(d_1, d_2\) are fixed regular elements in \((A, +)\) and \(\gamma_1, \ldots, \gamma_m, \lambda_1, \ldots, \lambda_n\), are commuting automorphisms of the semigroup \((A, +)\).

**Corollary 2.6.** Let \((A, F)\) be a regular medial algebra, then there exists a commutative semigroup \((A, +)\), such that every operation \(f \in F\) has the following representation:

\[ f(x_1, \ldots, x_m) = \gamma_1 x_1 + \cdots + \gamma_m x_m + d, \]

where \(d\) is a fixed regular element in \((A, +)\) and \(\gamma_1, \ldots, \gamma_m\) are commuting automorphisms of the semigroup \((A, +)\).

**Corollary 2.7.** If \((Q, f)\) is a medial \(n\)-ary quasigroup, then there exists an abelian group, \((Q, +)\), such that

\[ f(x_1, \ldots, x_m) = \alpha_1 x_1 + \cdots + \alpha_m x_m + d, \]

where \(\alpha_i \in \text{Aut}(Q, +)\), are pairwise commute, and \(d \in Q\).

There exist various algebraic characterizations of different classes of \(n\)-ary operations (see, for instance, \([4]\)).

**Theorem 2.8.** Let \((A, f)\) be a paramedial \(n\)-ary groupoid such that \(A\) contains an \(n\)-ary regular subgroupoid, then there exists a commutative semigroup with unit element \((A, +)\), such that

\[ f(x_1, \ldots, x_n) = \gamma_1 x_1 + \cdots + \gamma_n x_n + d, \]

where, \(d\) is a fixed regular element in \((A, +)\) and \(\gamma_1, \ldots, \gamma_n\), are automorphisms of the semigroup \((A, +)\), \(n \geq 2\). Moreover: \(\gamma_i \gamma_j = \gamma_{n-j+1} \gamma_{n-i+1}\), for \(n > 2\), \(i, j = 1, \ldots, n\), and for \(n = 2\) we have: \(\gamma_1^2 = \gamma_2^2\).

**Corollary 2.9.** Let \((Q, f)\) be a paramedial \(n\)-ary quasigroup, then there exists an abelian group \((Q, +)\), such that

\[ f(x_1, \ldots, x_n) = \gamma_1 x_1 + \cdots + \gamma_n x_n + d, \]

where, \(d\) is a fixed element in \((Q, +)\) and \(\gamma_1, \ldots, \gamma_n\), are automorphisms of the abelian group \((Q, +)\), \(n \geq 2\). Moreover: \(\gamma_i \gamma_j = \gamma_{n-j+1} \gamma_{n-i+1}\), for \(n > 2\), \(i, j = 1, \ldots, n\), and for \(n = 2\) we have: \(\gamma_1^2 = \gamma_2^2\).

**Theorem 2.10.** Let \((A, F)\) be a paramedial algebra with the regular idempotent element \(e\), then there exists a commutative semigroup \((A, +)\) with the unit element \(e\), such that every operation \(f \in F\) has the following linear representation

\[ f(x_1, \ldots, x_m) = \alpha_1 x_1 + \cdots + \alpha_m x_m, \]

where \(\alpha_1, \ldots, \alpha_m\), are pairwise commuting automorphisms of \((A, +)\), \(m \geq 2\).

**Theorem 2.11.** Let \((A, f, g)\) be a regular paramedial algebra with \(m\)-ary operation \(f\) and \(n\)-ary operation \(g\), then there is a commutative semigroup with unit element \((A, +)\), such that

\[ f(x_1, \ldots, x_m) = \gamma_1 x_1 + \cdots + \gamma_m x_m + d_1, \]
\[ g(x_1, \ldots, x_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n + d_2, \]

where, \(d_1, d_2\) are fixed regular elements in \((A, +)\) and \(\gamma_1, \ldots, \gamma_m, \lambda_1, \ldots, \lambda_n\), are commuting automorphisms of the semigroup \((A, +)\).

**Theorem 2.12.** Let \((Q, F)\) is a binary paramedial algebra with quasigroup operations, then there exists an abelian group \((Q, +)\), such that every operation, \(f_i \in F\), is represented by the following rule:

\[ f_i(x, y) = \varphi_i(x) + \psi_i(y) + c_i, \]
where $c_i \in \mathbb{Q}$ and $\varphi_i, \psi_i \in \text{Aut}(\mathbb{Q}, +)$, such that: $\varphi_i \varphi_j = \psi_j \psi_i$, $\varphi_i \psi_j = \psi_j \varphi_i$. The group, $(\mathbb{Q}, +)$, is unique up to isomorphisms.

**Theorem 2.13.** Let $(\mathbb{Q}, F)$ is a binary co-medial algebra with quasigroup operations, then there exists an abelian group $(\mathbb{Q}, +)$, such that every operation, $f_i \in F$, is represented by the following rule:

$$f_i(x, y) = \varphi_i(x) + \psi_i(y) + c_i,$$

where $c_i \in \mathbb{Q}$ and $\varphi_i, \psi_i \in \text{Aut}(\mathbb{Q}, +)$, such that $\varphi_i \varphi_j = \psi_j \psi_i$. The group, $(\mathbb{Q}, +)$, is unique up to isomorphisms.

**Theorem 2.14.** Let $(\mathbb{Q}, F)$ is a binary co-paramedial algebra with quasigroup operations, then there exists an abelian group $(\mathbb{Q}, +)$, such that every operation, $f_i \in F$, is represented by the following rule:

$$f_i(x, y) = \varphi_i(x) + \psi_i(y) + c_i,$$

where $c_i \in \mathbb{Q}$ and $\varphi_i, \psi_i \in \text{Aut}(\mathbb{Q}, +)$, such that $\varphi_i \varphi_j = \psi_j \psi_i$. The group, $(\mathbb{Q}, +)$, is unique up to isomorphisms.

Further description of the contents of this section are available in [3].

**References**


