An elementary proof of the completeness of the Łukasiewicz axioms

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The main aim of this talk is twofold. Firstly, to present an elementary method based on Farkas’ lemma for rationals how to embed any finite partial subalgebra of a linearly ordered MV-algebra into \( Q \ [0; 1] \) and then to establish a new elementary proof of the completeness of the Łukasiewicz axioms for which the MV-algebras community has been looking for a long time. Secondly, to present a direct proof of Di Nola’s representation Theorem for MV-algebras and to extend his results to the restriction of the standard MV-algebra on rational numbers.

1 Introduction

The representation theory of MV-algebras is based on Chang’s representation Theorem \[4\], McNaughton’s Theorem and Di Nola’s representation Theorem \[5\]. Chang’s representation Theorem yields a subdirect representation of all MV-algebras via linearly ordered MV-algebras. McNaughton’s Theorem characterizes free MV-algebras as algebras of continuous, piece-wise linear functions with integer coefficients on \([0, 1]\). Finally, Di Nola’s representation Theorem describes MV-algebras as sub-algebras of algebras of functions with values into a non-standard ultrapower of the MV-algebra \([0, 1]\).

The main motivation for our paper comes from the fact that although the proofs of both Chang’s representation Theorem \[4\] and McNaughton’s Theorem are of algebraic nature the proof of Di Nola’s representation Theorem is based on model-theoretical considerations. We give a simple, purely algebraic, proof of it and its variants based on the Farkas’ Lemma for rationals \[6\] and General finite embedding theorem \[3\].

1.1 Generalized finite embedding theorem

By an ultrafilter on a set \( I \) we mean an ultrafilter of the Boolean algebra \( \mathcal{P}(I) \) of the subsets of \( I \).

Let \( \{A_i; i \in I\} \) be a system of algebras of the same type \( F \) for \( i \in I \). We denote for any \( x, y \in \prod_{i \in I} A_i \) the set

\[
[x = y] = \{j \in I; x(j) = y(j)\}.
\]

If \( F \) is a filter of \( \mathcal{P}(I) \) then the relation \( \theta_F \) defined by

\[
\theta_F = \{(x, y) \in (\prod_{i \in I} A_i)^2; [x = y] \in F\}
\]

is a congruence on \( \prod_{i \in I} A_i \). For an ultrafilter \( U \) of \( \mathcal{P}(I) \), an algebra

\[
(\prod_{i \in I} A_i)/U := (\prod_{i \in I} A_i)/\theta_U
\]
is said to be an ultraproduct of algebras \( \{ A_i : i \in I \} \). Any ultraproduct of an algebra \( A \) is called an ultrapower of \( A \). The class of all ultraproducts (products, isomorphic images) of algebras from some class of algebras \( K \) is denoted by \( P_U(K) \) (\( P(K), I(K) \)). The class of all finite algebras from some class of algebras \( K \) is denoted by \( K_{\text{Fin}} \).

**Definition 1.** Let \( A = (A, F) \) be a partial algebra and \( X \subseteq A \). Denote the partial algebra \( A|_X = (X, F) \), where for any \( f \in F_n \) and all \( x_1, \ldots, x_n \in X \), \( f^{A|_X}(x_1, \ldots, x_n) \) is defined if and only if \( f^A(x_1, \ldots, x_n) \in X \) holds. Moreover, then we put

\[
f^{A|_X}(x_1, \ldots, x_n) := f^A(x_1, \ldots, x_n).
\]

**Definition 2.** An algebra \( A = (A, F) \) satisfies the general finite embedding property for the class \( K \) of algebras of the same type if for any finite subset \( X \subseteq A \) there are an (finite) algebra \( B \in K_E \) and an embedding \( \rho : A|_X \hookrightarrow B \), i.e. an injective mapping \( \rho : X \rightarrow B \) satisfying the property \( \rho(f^{A|_X}(x_1, \ldots, x_n)) = f^B(\rho(x_1), \ldots, \rho(x_n)) \) if \( x_1, \ldots, x_n \in X, f \in F_n \) and \( f^{A|_X}(x_1, \ldots, x_n) \) is defined.

Finite embedding property is usually denoted by (FEP). Note also that a quasivariety \( K \) has the FEP if and only if \( K = \text{ISP}_U(K_{\text{Fin}}) \) (see [2, Theorem 1.1] or [1]).

**Theorem 1.** [3, Theorem 6] Let \( A = (A, F) \) be a algebra and let \( K \) be a class of algebras of the same type. If \( A \) satisfies the general finite embedding property for \( K \) then \( A \in \text{ISP}_U(K) \).

**Theorem 2.** [3, Theorem 7] Let \( A = (A, F) \) be an algebra such that \( F \) is finite and let \( K \) be a class of algebras of the same type. If \( A \in \text{ISP}_U(K) \) then \( A \) satisfies the general finite embedding property for \( K \).

### 1.2 Farkas’ lemma

Let us recall the original formulation of Farkas’ lemma [6, 7] on rationals:

**Theorem 3** (Farkas’ lemma). Given a matrix \( A \in \mathbb{Q}^{m \times n} \) and \( c \) a column vector in \( \mathbb{Q}^m \), then there exists a column vector \( x \in \mathbb{Q}^n \), \( x \geq 0_n \) and \( A \cdot x = c \) if and only if, for all row vectors \( y \in \mathbb{Q}^m \), \( y \cdot A \geq 0_m \) implies \( y \cdot c \geq 0 \).

In what follows, we will use the following equivalent formulation:

**Theorem 4** (Theorem of alternatives). Let \( A \) be a matrix in \( \mathbb{Q}^{m \times n} \) and \( b \) a column vector in \( \mathbb{Q}^n \). The system \( A \cdot x \leq b \) has no solution if and only if there exists a row vector \( \lambda \in \mathbb{Q}^m \) such that \( \lambda \geq 0_m \), \( \lambda \cdot A = 0_n \) and \( \lambda \cdot b < 0 \).

### 2 The Embedding Lemma

In this section, we use the Farkas’ lemma on rationals to prove that any finite partial subalgebra of a linearly ordered MV-algebra can be embedded into \( \mathbb{Q} \cap [0, 1] \) and hence into the finite MV-chain \( L_k \subseteq [0, 1] \) for a suitable \( k \in \mathbb{N} \).

**Lemma 1.** Let \( M = (M; \oplus, \neg, 0) \) be a linearly ordered MV-algebra, \( X \subseteq M \setminus \{0\} \) be a finite subset. Then there is a rationally valued map \( s : X \cup \{0\} \rightarrow [0, 1] \cap \mathbb{Q} \) such that

1. \( s(0) = 0, s(1) = 1 \),

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2. if \( x, y, x \oplus y \in X \cup \{0, 1\} \) such that \( x \leq \neg y \) and \( x, y \in X \cup \{0, 1\} \) then \( s(x \oplus y) = s(x) + s(y) \).

3. if \( x \in X \) then \( s(x) > 0 \).

**Lemma 2 (Embedding Lemma).** Let us have a linearly ordered MV-algebra \( \mathbf{M} = (\mathbb{M}; \oplus, \neg, 0) \) and let \( X \subseteq \mathbb{M} \) be a finite set. Then there exists an embedding \( f : X \hookrightarrow \mathbb{L}_k \), where \( X \) is a partial MV-algebra obtained by the restriction of \( \mathbf{M} \) to the set \( X \) and \( \mathbb{L}_k \subseteq [0, 1] \) is the linearly ordered finite MV-algebra on the set \( \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\} \).

**3 Extensions of Di Nola’s Theorem**

In this section, we are going to show Di Nola’s representation Theorem and its several variants not only via standard MV-algebra \([0, 1]\) but also via its rational part \( \mathbb{Q} \cap [0, 1] \) and finite MV-chains. To prove it, we use the Embedding Lemma obtained in the previous section. First, we establish the FEP for linearly ordered MV-algebras.

**Theorem 5.**

1. The class \( \mathcal{LMV} \) of linearly ordered MV-algebras has the FEP.

2. The class \( \mathcal{MV} \) of MV-algebras has the FEP.

Note that the part (1) of the preceding theorem for subdirectly irreducible MV-algebras can be easily deduced from the result that the class of subdirectly irreducible Wajsberg hoops has the FEP (see \[1\, Theorem 3.9\]). The well-known part (2) then follows from \[1\, Lemma 3.7, Theorem 3.9\]. We are now ready to establish a variant of Di Nola’s representation Theorem for finite MV-chains (finite MV-algebras).

**Theorem 6.**

1. Any linearly ordered MV-algebra can be embedded into an ultraprodct of finite MV-chains.

2. Any MV-algebra can be embedded into a product of ultraprodcts of finite MV-chains.

3. Any MV-algebra can be embedded into an ultraprodct of finite MV-algebras (which are embeddable into powers of finite MV-chains).

The next two theorems cover Di Nola’s representation Theorem and its respective variants both for rationals and reals.

**Theorem 7.**

1. Any linearly ordered MV-algebra can be embedded into an ultrapower of \( \mathbb{Q} \cap [0, 1] \).

2. Any MV-algebra can be embedded into a product of ultrapowers of \( \mathbb{Q} \cap [0, 1] \).

3. Any MV-algebra can be embedded into an ultrapower of the countable power of \( \mathbb{Q} \cap [0, 1] \).

4. Any MV-algebra can be embedded into an ultraprodct of finite powers of \( \mathbb{Q} \cap [0, 1] \).

**Theorem 8.**

1. Any linearly ordered MV-algebra can be embedded into an ultrapower of \([0, 1] \).

2. Any MV-algebra can be embedded into a product of ultrapowers of \([0, 1] \).

3. Any MV-algebra can be embedded into an ultrapower of the countable power of \([0, 1] \).

4. Any MV-algebra can be embedded into an ultraprodct of finite powers of \([0, 1] \).

**Proof.** (1)-(4) It is a corollary of Theorem 6.
References