An elementary proof of the completeness of the Łukasiewicz axioms

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The main aim of this talk is twofold. Firstly, to present an elementary method based on Farkas' lemma for rationals how to embed any finite partial subalgebra of a linearly ordered MV-algebra into Q [0; 1] and then to establish a new elementary proof of the completeness of the Lukasiewicz axioms for which the MV-algebras community has been looking for a long time. Secondly, to present a direct proof of Di Nola's representation Theorem for MV-algebras and to extend his results to the restriction of the standard MV-algebra on rational numbers.

1 Introduction

The representation theory of MV-algebras is based on Chang's representation Theorem [4], McNaughton's Theorem and Di Nola's representation Theorem [5]. Chang's representation Theorem yields a subdirect representation of all MV-algebras via linearly ordered MV-algebras. McNaughton's Theorem characterizes free MV-algebras as algebras of continuous, piece-wise linear functions with integer coefficients on $[0, 1]$. Finally, Di Nola's representation Theorem describes MV-algebras as sub-algebras of algebras of functions with values into a non-standard ultrapower of the MV-algebra $[0, 1]$.

The main motivation for our paper comes from the fact that although the proofs of both Chang's representation Theorem [4] and McNaughton's Theorem are of algebraic nature the proof of Di Nola's representation Theorem is based on model-theoretical considerations. We give a simple, purely algebraic, proof of it and its variants based on the Farkas' Lemma for rationals [6] and General finite embedding theorem [3].

1.1 Generalized finite embedding theorem

By an *ultrafilter* on a set I we mean an ultrafilter of the Boolean algebra $\mathcal{P}(I)$ of the subsets of I.

Let $\{\mathbf{A}_i;\ i \in I\}$ be a system of algebras of the same type F for $i \in I$. We denote for any $x, y \in \prod_{i\in I} A_i$ the set

$$[\![x = y]\!] = \{j \in I; x(j) = y(j)\}.$$

If F is a filter of $\mathcal{P}(I)$ then the relation θ_F defined by

$$\theta_F = \{\langle x, y\rangle \in (\prod_{i\in I} A_i)^2; [\![x = y]\!] \in F\}$$

is a congruence on $\prod_{i\in I} \mathbf{A}_i$. For an ultrafilter U of $\mathcal{P}(I)$, an algebra

$$(\prod_{i\in I} \mathbf{A}_i)/U := (\prod_{i\in I} \mathbf{A}_i)/\theta_U$$

is said to be an *ultraproduct* of algebras $\{\mathbf{A}_i;\ i \in I\}$. Any ultraproduct of an algebra $\mathbf{A}$ is called an ultrapower of $\mathbf{A}$. The class of all ultraproducts (products, isomorphic images) of algebras from some class of algebras $\mathcal{K}$ is denoted by $\mathrm{P_U}(\mathcal{K})$ ($\mathrm{P}(\mathcal{K})$, $\mathrm{I}(\mathcal{K})$). The class of all finite algebras from some class of algebras $\mathcal{K}$ is denoted by $\mathcal{K}_{Fin}$.

Definition 1. *Let $\mathbf{A} = (A, \mathrm{F})$ be a partial algebra and $X \subseteq A$. Denote the partial algebra $\mathbf{A}|_X = (X, \mathrm{F})$, where for any $f \in F_n$ and all $x_1, \dots, x_n \in X$, $f^{\mathbf{A}|_X}(x_1, \dots, x_n)$ is defined if and only if $f^{\mathbf{A}}(x_1, \dots, x_n) \in X$ holds. Moreover, then we put*

$$f^{\mathbf{A}|_X}(x_1, \dots, x_n) := f^{\mathbf{A}}(x_1, \dots, x_n).$$

Definition 2. An algebra $\mathbf{A} = (A, \mathrm{F})$ satisfies *the general finite embedding* (*finite embedding property*) *property for the class* $\mathcal{K}$ of algebras of the same type if for any finite subset $X \subseteq A$ there are an (finite) algebra $\mathbf{B} \in \mathcal{K}_{\mathrm{E}}$ and an embedding $\rho : \mathbf{A}|_X \hookrightarrow \mathbf{B}$, i.e. an injective mapping $\rho : X \to B$ satisfying the property $\rho(f^{\mathbf{A}|_X}(x_1, \dots, x_n)) = f^{\mathbf{B}}(\rho(x_1), \dots, \rho(x_n))$ if $x_1, \dots, x_n \in X$, $f \in \mathrm{F}_n$ and $f^{\mathbf{A}|_X}(x_1, \dots, x_n)$ is defined.

Finite embedding property is usually denoted by (FEP). Note also that a quasivariety $\mathcal{K}$ has the FEP if and only if $\mathcal{K} = \mathrm{ISPP_U}(\mathcal{K}_{Fin})$ (see [2, Theorem 1.1] or [1]).

Theorem 1. [3, Theorem 6] *Let $\mathbf{A} = (A, \mathrm{F})$ be a algebra and let $\mathcal{K}$ be a class of algebras of the same type. If $\mathbf{A}$ satisfies the general finite embedding property for $\mathcal{K}$ then $\mathbf{A} \in \mathrm{ISP_U}(\mathcal{K})$.*

Theorem 2. [3, Theorem 7] *Let $\mathbf{A} = (A, \mathrm{F})$ be an algebra such that F is finite and let $\mathcal{K}$ be a class of algebras of the same type. If $\mathbf{A} \in \mathrm{ISP_U}(\mathcal{K})$ then $\mathbf{A}$ satisfies the general finite embedding property for $\mathcal{K}$.*

1.2 Farkas' lemma

Let us recall the original formulation of Farkas' lemma [6, 7] on rationals:

Theorem 3 (Farkas' lemma)**.** *Given a matrix A in $\mathbb{Q}^{m\times n}$ and $\boldsymbol{c}$ a column vector in $\mathbb{Q}^m$, then there exists a column vector $\boldsymbol{x} \in \mathbb{Q}^n$, $\boldsymbol{x} \geq \mathbf{0}_n$ and $A \cdot \boldsymbol{x} = \boldsymbol{c}$ if and only if, for all row vectors $\boldsymbol{y} \in \mathbb{Q}^m$, $\boldsymbol{y} \cdot A \geq \mathbf{0}_m$ implies $\boldsymbol{y} \cdot \boldsymbol{c} \geq 0$.*

In what follows, we will use the following equivalent formulation:

Theorem 4 (Theorem of alternatives)**.** *Let A be a matrix in $\mathbb{Q}^{m\times n}$ and $\boldsymbol{b}$ a column vector in $\mathbb{Q}^n$. The system $A \cdot \boldsymbol{x} \leq \boldsymbol{b}$ has no solution if and only if there exists a row vector $\boldsymbol{\lambda} \in \mathbb{Q}^m$ such that $\boldsymbol{\lambda} \geq \mathbf{0}_m$, $\boldsymbol{\lambda} \cdot A = \mathbf{0}_n$ and $\boldsymbol{\lambda} \cdot \boldsymbol{b} < 0$.*

2 The Embedding Lemma

In this section, we use the Farkas' lemma on rationals to prove that any finite partial subalgebra of a linearly ordered MV-algebra can be embedded into $\mathbb{Q} \cap [0, 1]$ and hence into the finite MV-chain $\mathbf{L}_k \subseteq [0, 1]$ for a suitable $k \in \mathbb{N}$.

Lemma 1. *Let $\mathbf{M} = (M; \oplus, \neg, 0)$ be a linearly ordered MV-algebra, $X \subseteq M \setminus \{0\}$ be a finite subset. Then there is a rationally valued map $s : X \cup \{0, 1\} \longrightarrow [0, 1] \cap \mathbb{Q}$ such that*

1. $s(0) = 0$, $s(1) = 1$,

2. *if* $x, y, x \oplus y \in X \cup \{0,1\}$ *such that* $x \leq \neg y$ *and* $x, y \in X \cup \{0,1\}$ *then* $s(x \oplus y) = s(x) + s(y)$.

3. *if* $x \in X$ *then* $s(x) > 0$.

Lemma 2 (**Embedding Lemma**). *Let us have a linearly ordered MV-algebra* $\mathbf{M} = (M; \oplus, \neg, 0)$ *and let* $X \subseteq M$ *be a finite set. Then there exists an embedding* $f : \mathbf{X} \hookrightarrow \mathbf{L}_k$, *where* $\mathbf{X}$ *is a partial MV-algebra obtained by the restriction of* $\mathbf{M}$ *to the set* X *and* $\mathbf{L}_k \subseteq [0,1]$ *is the linearly ordered finite MV-algebra on the set* $\{0, \frac{1}{k}, \frac{2}{k}, \cdots, 1\}$.

3 Extensions of Di Nola's Theorem

In this section, we are going to show Di Nola's representation Theorem and its several variants not only via standard MV-algebra $[0,1]$ but also via its rational part $\mathbb{Q} \cap [0,1]$ and finite MV-chains. To prove it, we use the Embedding Lemma obtained in the previous section. First, we establish the FEP for linearly ordered MV-algebras.

Theorem 5. *1. The class* $\mathcal{LMV}$ *of linearly ordered MV-algebras has the FEP.*

2. The class $\mathcal{MV}$ *of MV-algebras has the FEP.*

Note that the part (1) of the preceding theorem for subdirectly irreducible MV-algebras can be easily deduced from the result that the class of subdirectly irreducible Wajsberg hoops has the FEP (see [1, Theorem 3.9]). The well-known part (2) then follows from [1, Lemma 3.7,Theorem 3.9]. We are now ready to establish a variant of Di Nola's representation Theorem for finite MV-chains (finite MV-algebras).

Theorem 6. *1. Any linearly ordered MV-algebra can be embedded into an ultraproduct of finite MV-chains.*

2. Any MV-algebra can be embedded into a product of ultraproducts of finite MV-chains.

3. Any MV-algebra can be embedded into an ultraproduct of finite MV-algebras (which are embeddable into powers of finite MV-chains).

The next two theorems cover Di Nola's representation Theorem and its respective variants both for rationals and reals.

Theorem 7. *1. Any linearly ordered MV-algebra can be embedded into an ultrapower of* $\mathbb{Q} \cap [0,1]$.

2. Any MV-algebra can be embedded into a product of ultrapowers of $\mathbb{Q} \cap [0,1]$.

3. Any MV-algebra can be embedded into an ultrapower of the countable power of $\mathbb{Q} \cap [0,1]$.

4. Any MV-algebra can be embedded into an ultraproduct of finite powers of $\mathbb{Q} \cap [0,1]$.

Theorem 8. *1. Any linearly ordered MV-algebra can be embedded into an ultrapower of* $[0,1]$.

2. Any MV-algebra can be embedded into a product of ultrapowers of $[0,1]$.

3. Any MV-algebra can be embedded into an ultrapower of the countable power of $[0,1]$.

4. Any MV-algebra can be embedded into an ultraproduct of finite powers of $[0,1]$.

Proof. (1)-(4) It is a corollary of Theorem 6. □

References

[1] Blok, W., Ferreirim, I.: On the structure of hoops, Algebra Universalis 43, 233-257 (2000).

[2] Blok, W., van Alten, C. J.: On the finite embeddability property for residuated ordered groupoids, Trans. Amer. Math. Soc. 357, 4141–4157 (2005).

[3] Botur, M: A non-associative generalization of Hájeks BL-algebras, Fuzzy Sets and Systems 178, 24–37 (2011).

[4] Chang, C.C.: Algebraic analysis of many valued logics, Trans. Amer. Math. Soc. 88, 467–490 (1958).

[5] Di Nola, A.: Representation and reticulation by quotients of MV-algebras, Ricerche di Matematica XL, 291–297 (1991).

[6] Farkas, G.: Über die Theorie der einfachen Ungleichungen, Journal für die Reine und Angewandte Mathematik 124, 1–27 (1902).

[7] Schrijver, A.: Theory of linear and integer programming, Wiley-Interscience series in discrete mathematics and optimization, John Wiley & sons (1998).