Derivational modal logic of real line with difference modality

Kudinov Andrey

Institute for Information Transmission Problems, Russian Academy of Sciences
National Research University Higher School of Economics, Moscow, Russia
Moscow Institute of Physics and Technology
kudinov@iitp.ru

Abstract

We study derivational modal logic of real line with difference modality and prove that it has finite model property but does not have finite axiomatization.

Definition 1. Formulas are constructed from set of variables Prop, constant ⊥ and connective → and modal operators □ and [≠]. The other connectives (¬, ∨, ∧) considered as short-cuts. Dual modalities □ and ⟨[≠]⟩ are expressible in the following way

□A ⇔ ¬□¬A, ⟨[≠]⟩A ⇔ ¬[≠]¬A.

We will also use the following short-cuts [∀]A ⇔ [≠]A ∧ A, ⊞A = □A ∧ A.

Definition 2. Topological model is a pair M = (X, θ), where X is a topological space and θ : Prop → 2X is a valuation on X. Truth of a formula A at a point x ∈ X (M, x ⊨ A) is defined by induction as usual:

1) M, x ⊨ p ⇐⇒ x ∈ θ(p)
2) M, x ⊨ A → B ⇐⇒ (M, x ⊨ A ⇒ M, x ⊨ B)
3) M, x ⊨ □A ⇐⇒ ∃U(x)∀y ∈ U(x) (y ≠ x ⇒ M, y ⊨ A)
4) M, x ⊨ [≠]A ⇐⇒ ∀y (y ≠ x ⇒ M, y ⊨ A)

where p ∈ Prop and U(x) is a neighbourhood of x. Formula is called dd-valid in a space X (or class of spaces C) if it is true at all points in all models on X (in all spaces from C).

Definition 3. Kripke frame F is a triple (W, R, RD), where W ≠ ∅ and R, RD ⊆ W × W.

Definition 4. Kripke model on a (Kripke) frame F is a pair M = (F, θ), where θ : Prop → 2W is a valuation on F. The truth of a formula at a point in a Kripke model defines as usual in particular

M, x ⊨ □φ ⇐⇒ ∀y (xRy ⇒ M, y ⊨ φ),
M, x ⊨ [≠]φ ⇐⇒ ∀y (xRDy ⇒ M, y ⊨ φ).

Definition 5. A (normal) 2-modal logic is a set of modal formulas containing the classical tautologies, axiom □(p → q) → (□p → □q) and the same for [≠], and closed under the standard inference rules: Modus Ponens (A, A → B/B), Necessitation (A/□A), and Substitution (A(pj)/A(B)).

K2 stands for the minimal 2-modal logic. An 2-modal logic containing a certain 2-modal logic L is called an extension of L, or a L-logic. The minimal L-logic containing a set of 2-modal formulas Γ is denoted by L + Γ.

The work on this paper is patially sponsored by RFBR grants 11-01-00958-a and 11-01-93107-a


136
Consider the following axioms

\[
\begin{align*}
(B_D) & \quad p \rightarrow [\neq](\neq)p \\
(4_D^0) & \quad (p \land [\neq]p) \rightarrow [\neq][\neq]p \\
(4_\Box) & \quad [\neq]p \rightarrow \Box\Box p \\
(4_\Diamond) & \quad \Box p \rightarrow \Box\Box p \\
(D_\Box) & \quad [\neq]p \rightarrow \Box p \\
(AT_1) & \quad [\neq]p \rightarrow [\neq]\Box p \\
(DS) & \quad \Diamond \top \\
(AC) & \quad [\forall](\Box p \lor \Box \neg p) \rightarrow [\forall]p \lor [\forall]\neg p \\
(Ku_2) & \quad \Box \bigvee_{k=0}^2 p \lor \bigvee_{k=0}^2 \neg p,
\end{align*}
\]

where \(Q_1 = q_1 \land q_2\), \(Q_2 = q_1 \land \neg q_2\) and \(Q_3 = \neg q_1\).

We introduce the following logics

\[
\begin{align*}
D_4 & = K \{4_\Box, D_\Box\}, \\
K4^0D & = K_2 \{B_D, 4_D^0, D_\Box, 4_\Box\}, \\
LC_2 & = K4^0D \cup \{4_\Box, AT_1, DS, AC, Ku_2\}.
\end{align*}
\]

Logic of a class \(C\) of Kripke frames is the set of all formulas valid in all frames from \(C\) (notation: \(L(C)\)). A frame \(F\) called an \(-frame if for any connected neighbourhood \(\mathcal{U}\) of any point \(x\). Locally connected \(X\) is called \(-component if for any connected neighbourhood \(U\) of any point \(x\) contains at most \(n\) connected components.

The following correspondences is well-known ([1], [5], [3]).

<table>
<thead>
<tr>
<th>Axioms</th>
<th>Property of topological space</th>
</tr>
</thead>
<tbody>
<tr>
<td>(DS)</td>
<td>density-in-itself</td>
</tr>
<tr>
<td>(4_\Box)</td>
<td>(T_d)</td>
</tr>
<tr>
<td>(AT_1)</td>
<td>(T_1)</td>
</tr>
<tr>
<td>(Ku_2)</td>
<td>locally 2-componentness</td>
</tr>
<tr>
<td>(AC)</td>
<td>connectedness</td>
</tr>
</tbody>
</table>

Real line \(\mathbb{R}\) satisfy these properties so \(Ld_{\neq}(\mathbb{R}) \supseteq LC_2\). But in fact \(Ld_{\neq}(\mathbb{R}) \neq LC_2\) (see Theorem 4).

Let \(F = (W, R, R_D)\) be an \(K4^0D\)-frame. Then \(R\) is weakly transitive relation and \(R_D \cup Id_W\) is universal. Put \(\hat{R} = R \circ R^{-1}\); \(\hat{R}\) is the reflexive transitive closer of \(R\). Then \(\hat{R}\) is an equivalence. Frame \(F\) is called connected, if \(\forall x \forall y (x \hat{R} y). \ R^* \equiv \hat{R}|_{W^{-}}, \) where \(W^- = \{w \in W \ | \ w R_D w\}. \ F\) locally 2-component, if for each \(x\) \(R(x)\) intersects with at most 2 \(R^*\)-classes.

The following correspondences can be found in [3].

<table>
<thead>
<tr>
<th>Axioms</th>
<th>Property of Kripke frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>(DS)</td>
<td>(\forall x (R(x) \cap R_D(x) \neq \emptyset))</td>
</tr>
<tr>
<td>(4_\Box)</td>
<td>transitivity of (R)</td>
</tr>
<tr>
<td>(AT_1)</td>
<td>(\forall x (\neg x R_D x \Rightarrow \neg \exists y (y R x \land y \neq x)))</td>
</tr>
<tr>
<td>(Ku_2)</td>
<td>locally 2-componentness</td>
</tr>
<tr>
<td>(AC)</td>
<td>connectedness</td>
</tr>
</tbody>
</table>
Lemma 2. [2] If $f : X \to F$ and $F$ is a finite $K^0$-frame then $Ld_{\not=} (X) \subseteq L(F)$.

Definition 7. Let $F = (W, R, R_D)$ be a $K^0$-frame and $X$ a topological space. An onto function $f : X \to F$ is called $dd$-morphism (notation: $f : X \to^{dd} F$) if for any $w \in W$: (1) $d f^{-1}(w) = f^{-1}(R^{-1}(w))$, (2) if $−wR_D w$ then $f^{-1}(w)$ is a one-element-set (where $d$ is the derivative operator on $X$).

Lemma 3. If $f : X \to^{dd} F$ and $F$ is a finite $K^0$-frame then $Ld_{\not=} (X) \subseteq L(F)$.

Proof. From left to right it was proved in [2].

From right to left we need to construct a $dd$-morphism. We use the result from from [4]. The author constructed a d-morphism from $\mathbb{R}$ onto any connected 2-component D-frame. Note that in this construction preimage of any irreflexive root point is a one-element-set.

Let $A = \{x \in W \mid \neg xRx\}$ and $B = \{x \in W \mid \neg \neg xR_D x\}$. Due to axiom $D \sqcup B \sqsubseteq A$. Since $F$ is finite assume that $C = A - B = \{c_1, \ldots, c_k\}$. If $C = \emptyset$ then $F' = F$.

Otherwise, we define $W' = W \cup C'$, where $C' = \{c'_1, \ldots, c'_k\}$ be a disjoint copy of $C$. $R'$ is an extension of $R$, such that $R'(c'_i) = R(c_i)$ and $R'_D$ is such that $R'_D \cup Id = W' \times W'$ and the only $R'_D$-irreflexive points are $A \cup C'$. Put $F' = (W', R', R'_D)$.

Let $g : W' \to W$ is identical on $W$ and for any $i \in \{1, \ldots, k\}$ $g(c'_i) = c_i$. It is easy to check that $g$ is a $p$-morphism. Now we use [4] and construct $d$-morphism $h : \mathbb{R} \to^d F'$ and we put $f = g \circ h$. We left to the reader to check that $f : \mathbb{R} \to^{dd} F$. \hfill \Box

This Lemma allow us to prove that $dd$-logic of $\mathbb{R}$ is not finitely axiomatizable and even stronger theorem. A logic is called $n$-axiomatizable is it has an axiomatization which uses only $n$ variables.

Theorem 4. Logic $Ld_{\not=} (\mathbb{R})$ is not $n$-axiomatizable for any $n$.

Basically the construction repeats the one from [2].

Let $C_E$ be the set of all finite $LC_2$-frames, such that $\Gamma(F)$ is traversable and $L_E = L(C_E)$.

By Lemmas 3 and 2 $Ld_{\not=} (\mathbb{R}) \subseteq L_E$. We can even prove

Theorem 5. Logics $Ld_{\not=} (\mathbb{R})$ and $L_E$ coincides.

This theorem can be proved using topofiltration — an analogue of epifiltration. We are filtering topological model and get a finite $LC_2$-frame $F$ such that $\Gamma(F)$ is traversable. Since the size of the frame has an upper bound

Corollary 6. Logic $Ld_{\not=} (\mathbb{R})$ is decidable.
References