From SAT to Maximum Independent Set: A New Approach to Characterize Tractable Classes

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Motivations & Objectives

- Characterizing new tractable classes remains important for both SAT and AI in general
- Cornerstone step towards understanding the practical effectiveness of SAT solvers

- Exploit the polynomial reducibility, one of the fundamental concepts in complexity theory
- to characterize new tractable classes in SAT thanks to tractability results obtained for other NP-Compete problems (e.g. maximum independent set problem)
Outline

Propositional Satisfiability (SAT) - Maximum Independent Set (MIS)

Tractability Results: from MIS to SAT

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Conclusion and perspectives
Conjunctive Normal Form (CNF) and SAT

- A conjunction of clauses:

\[
\text{clause} = (x_1 \lor \cdots \lor x_l) \land (y_1 \lor \cdots \lor y_m) \land (z_1 \lor \cdots \lor z_n) \cdots
\]

- Clause: a disjunction of literals \((x, \neg x)\)

- Example:

\[
\Phi = (p \lor \neg q \lor \neg r) \land (p \lor \neg q \lor s) \land p \land (r \lor \neg s)
\]

\[\mathcal{I}(p) = 1 \text{ and } \mathcal{I}(r) = 1 \text{ (Partial interpretation)}\]

Satisfiability: \(\exists \mathcal{I}, \mathcal{I}(\Phi) = 1\) (NP-complete [Cook 71])

- Tractable classes: 2-SAT, (Renamable) Horn, etc.
Boolean Satisfiability Problem (SAT)

- Spectacular progress $\rightarrow$ Modern SAT solvers
  - Application instances with millions of variables and clauses
- Many applications
  - Formal Verification
  - Planning
  - Bioinformatics
  - Cryptography
  - ...
- Around SAT
  - Max-SAT, (Weighted) Partial Max-SAT, QBF, ...
- CRIL Projects
  - Microsoft Research (UK): 2007-2012 (with Youssef Hamadi)
  - RATP (France): 2015-2017
  - ANR Project TUPLES "Tractability for Understanding and Pushing forward the Limits of Efficient Solvers", 2010-2014
Maximum Independent set problem

- Given an undirected graph $G = (V, E)$,
- An *independent set* of $G$ is a set of non-adjacent vertices.
- A *Maximum independent* is a sub-set $V' \subset V$ of maximum cardinality such that for all $u, v \in V'$, $(u, v) \notin E$. We note $\alpha(G)$ this maximum size.

- Maximum Independent Set Problem (MIS) : Given a graph $G$, find a maximum independent set of $G$ (NP-Hard)

- MIS tractable classes: *claw-free graphs*, perfect graphs
Tractable classes: claw-free graph

- A *claw-free graph* is a graph that does not have a *claw* as an induced subgraph.

- Finding a MIS in this class of graphs is tractable [Minty 80, Sbihi 80]
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Definition (PBN formula)

A PBN formula $\phi$ is a CNF formula where each clause is either a positive clause or a binary negative clause.

- Notations: $\text{Pos}(\phi)$ (resp. $\text{Neg}(\phi)$) to denote the set of its positive (resp. negative) clauses.

- Example:

$$\phi_{PBN} = \left\{ (p \lor q \lor r \lor s) \land (r \lor s \lor t) \land (\neg p \lor \neg v) \land (\neg q \lor \neg s) \right\}$$

- Checking the satisfiability of a PBN formula is NP-complete.
Transformation of CNF formula to PBN

- Associate to each negative literal $l$, a fresh variable $r_l$

- Replace each clause of the form:
  \[ p_1 \lor \cdots \lor p_m \lor \neg q_1 \lor \cdots \lor \neg q_n \]
  with
  \[ \{ p_1 \lor \cdots \lor p_m \lor r_{-q_1} \lor \cdots \lor r_{-q_n}, \neg q_1 \lor \neg r_{-q_1}, \ldots, \neg q_n \lor \neg r_{-q_n} \} \]

- Remark: the first clause can be obtained from several applications of the resolution rule on the last set of clauses

\[
\phi_{CNF} = \begin{cases} 
(p \lor q \lor \neg r) \land \\
(p \lor \neg s) 
\end{cases} 
\]

\[
\phi_{PBN} = \begin{cases} 
(p \lor q \lor t_{-r}) \land \\
(p \lor t_{-s}) \land \\
(\neg r \lor \neg t_{-r}) \land \\
(\neg s \lor \neg t_{-s}) 
\end{cases} 
\]
Definition (SPBN formula)
An SPBN formula $\phi$ is a PBN formula where, for all $c \in \text{Pos}(\phi)$, $|\mathcal{P}(\text{Pos}(\phi) \setminus \{c\}) \cap c| \leq 1$.

Example
$\phi_{\text{SPBN}} = (p \lor q) \land (p \lor r) \land (t \lor u) \land (\neg q \lor \neg r) \land (\neg p \lor \neg u)$

Definition (IPBN formula)
An IPBN formula $\phi$ is a PBN formula where, for all $c, c' \in \text{Pos}(\phi)$ with $c \neq c'$, $c \cap c' = \emptyset$.

Example
$\phi_{\text{IPBN}} = (p \lor q \lor v) \land (r \lor s \lor w) \land (t \lor u) \land (\neg p \lor \neg t) \land (\neg q \lor \neg r)$
A hierarchy of syntactic fragments of SAT

PBN – SAT

SPBN – SAT

IPBN – SAT

(NP – Complete)
Example of a IPBN formula

- The pigeon-hole principle $PHP^b_n$: mapping $b$ pigeons to $n$ holes, with $b > n$

- The pigeon-hole principle $php^b_n$ is expressed using the following IPBN formula:

\[ \bigvee_{j=1}^{n} p_{ij}, \; 1 \leq i \leq b \]  

(1)

\[ \neg p_{ij} \lor \neg p_{kj}, \; 1 \leq i < k \leq b \text{ and } 1 \leq j \leq n \]  

(2)

where $p_{ij} = 1$ iff the pigeon $i$ is put in the hole $j$.

- Each positive clause of $\phi_{php^b_n}$ does not share any literals with the other positive clauses.
PBN formula

- Given a PBN formula $\phi$, $\mathcal{R}(\phi)$ to denote the PBN formula
  $\phi \cup \{\neg p \lor \neg q \mid p \neq q \text{ and } \exists c \in \text{Pos}(\phi) \text{ s.t. } \{p, q\} \subseteq c\}$.

Proposition

Given an PBN formula $\phi$, $\phi$ is satisfiable iff $\mathcal{R}(\phi)$ is satisfiable.

- The previous proposition shows that we only need to satisfy one literal in each positive clause.
Tractable Classes in IPBN-SAT: transformation

$$\phi_{IPBN} = \left\{ \begin{array}{l}
(p \lor q \lor r) \land \\
(s \lor t) \land \\
(\neg q \lor \neg s) \land \\
(\neg r \lor \neg s) \land \\
(\neg r \lor \neg t) 
\end{array} \right.$$
Let $\phi$ be an IPBN formula. $G_{\phi} = (V, E)$ denotes the undirected graph of $\phi$. Where $V = \text{Var}(\phi)$ and $\{p, q\} \in E$ iff $p$ and $q$ are in the same clause.

**Theorem**

Given an IPBN formula $\phi$, $\phi$ is satisfiable iff $\alpha(G_{\phi}) \geq n$ where $n = |\text{Pos}(\phi)|$.

- It suffices to satisfy one literal by positive clause.
Tractable Classes in IPBN-SAT: recognition

- If $G_\phi$ is claw-free so the corresponding IPBN formula $\phi$ is claw-free.

**Theorem**

*(recognition) Checking whether a CNF formula is claw-free is tractable.*

- Checking whether the corresponding graph is claw-free is tractable
Theorem
*(satisfiability)* Checking the satisfiability of a claw-free IPBN formula is tractable.

- A direct consequence of the first theorem.
- MIS in the class of claw-free graphs is tractable.

Proposition
*The pigeon-hole principle is an instance of claw-free IPBN-SAT.*
Tractable Classes in SPBN-SAT: transformation

- Example: \( \phi = (p \lor q) \land (p \lor r) \land (\neg p \lor \neg u) \land (\neg q \lor \neg r) \)
- \( S(p, \phi) = \{ V^1_p, V^2_p \}, S(q, \phi) = \{ V_q \}, \ldots \)
- Edges are added between all occurrences of \( p \) (\( \{ v^1_p, v^2_p \} \)) and \( q \) (\( \{ v_q \} \)).
Theorem

Given an SPBN formula $\phi$, $\phi$ is satisfiable iff $\alpha(G_{\phi} \geq n$ where $n = |Pos(\phi)|$.

- Generalization of the first theorem.
- The occurrences of the positive literals are taken into account, since positive clauses may share literals.
Let $\phi$ be a PBN formula

We note $C(p, \phi) = \{ c \in \phi | p \in c \}$ the "cover" of $p$ in $\phi$

**Definition**

A CC-PBN formula $\phi$ is a PBN formula where, for all $p, q \in \text{Var}(\phi)$ with $p, q \in c$ for a clause $c \in \text{Pos}(\phi)$, we have $C(p, \text{Pos}(\phi)) \subseteq C(q, \text{Pos}(\phi))$ or $C(q, \text{Pos}(\phi)) \subseteq C(p, \text{Pos}(\phi))$. 
Tractable Classes in PBN-SAT: CC-PBN example

Example

$\phi$ is a CC-PBN formula:

$\phi = ([p] \lor [q] \lor r) \land$

$([p] \lor [q]) \land$

$([p] \lor t) \land$

$(\neg r \lor \neg t)$

Note: $C(q, Pos(\phi)) \subset C(p, Pos(\phi))$
Tractable Classes in PBN-SAT: CC-PBN

Proposition
Given an CC-PBN formula $\phi$, $\phi$ is satisfiable iff $R(\phi)$ is satisfiable.

Theorem
Given a CC-PBN formula $\phi$, $\phi$ is satisfiable iff $\alpha(G^\phi_\phi) \geq n$ where $n = |Pos(\phi)|$. 
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Characterization of the minimal models in CC-PBN formulæ using the MIS.

Definition
Let $\phi$ be a propositional formula and $I$ a model of $\phi$. Then $I$ is said to be a minimal model of $\phi$ if there is no model strictly smaller than $I$ w.r.t. $\preceq$.

$$\rightarrow I \preceq I' \text{ if } \{ p \in \mathcal{P}(\phi) | I(p) = 1 \} \subseteq \{ p \in \mathcal{P}(\phi) | I'(p) = 1 \}$$

Proposition
Let $\phi$ be a CC-PBN formula and $I$ a model of $\phi$. Then, $I$ is a minimal model of $\phi$ iff $I$ satisfies exactly one literal in each positive clause.

Theorem
Let $\phi$ be a CC-PBN formula and $I$ a Boolean interpretation of $\phi$. $I$ is a minimal model of $\phi$ iff $\alpha(G_\phi) = |\text{Pos}(\phi)|$ and $\bigcup_{p \in E} S(p, \phi)$ is a maximum independent set of $G_\phi$ where $E = \{ p \in \mathcal{P}(\phi) | I(p) = 1 \}$.
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- Cross-fertilization between MIS and SAT
- New tractable classes.
- Pigeonhole problem belongs to one of these classes: IPBN.

- Characterization of new tractable classes modulo occurrences of the variables in the positives clauses, e.g.: the Tovey classes.
- PBN is a suitable form for connecting tractables classes between CSP and SAT
Thank you.