# Coq without Casts : A complete proof of Coq Modulo Theory

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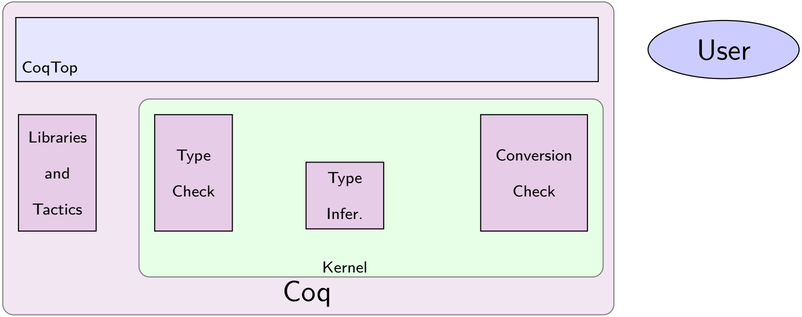
# Content

# Workflow of Coq



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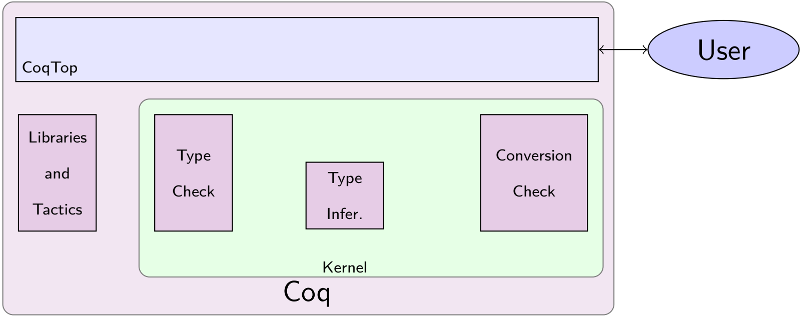

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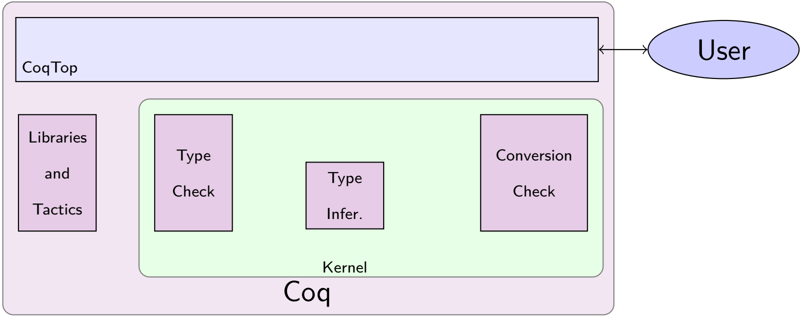


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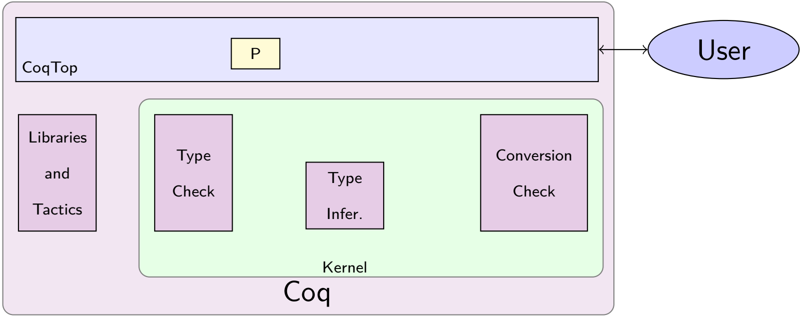


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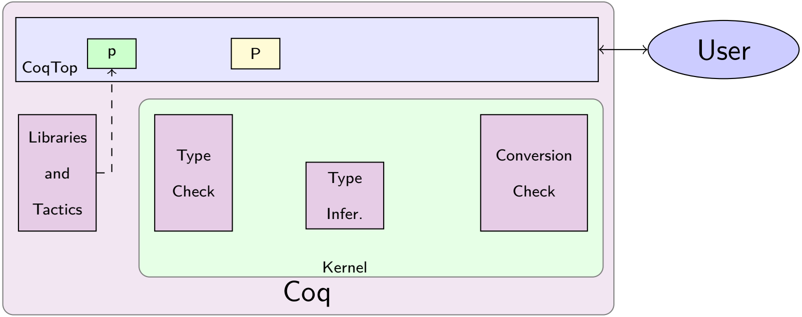


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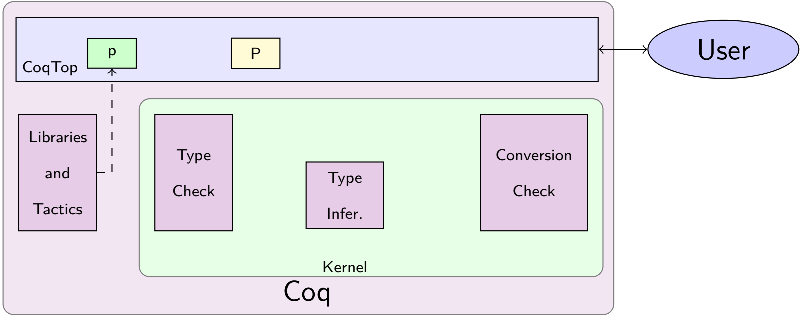


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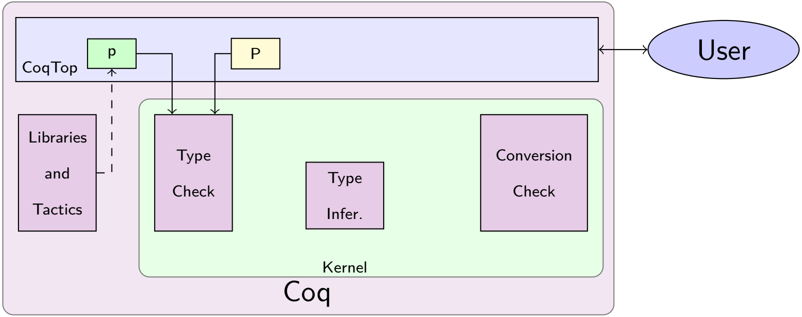


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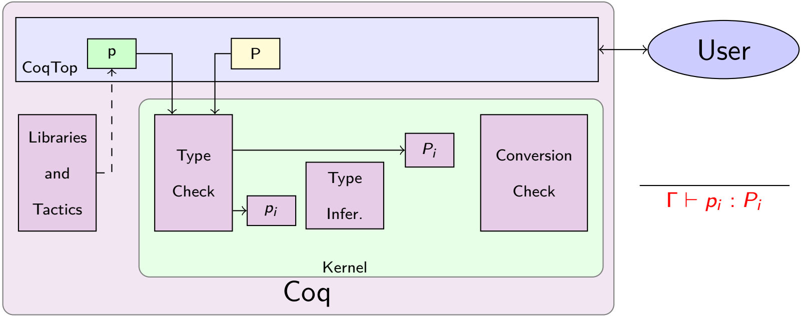


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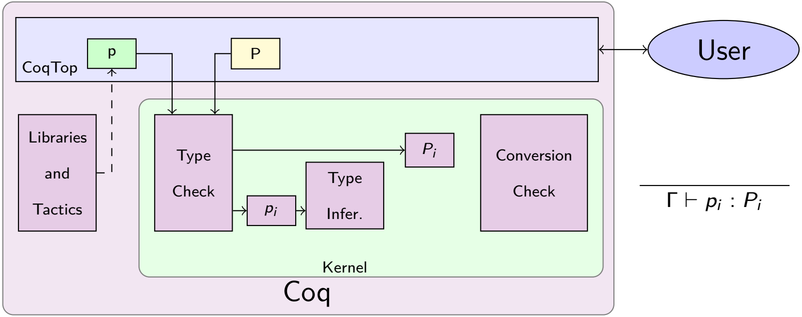


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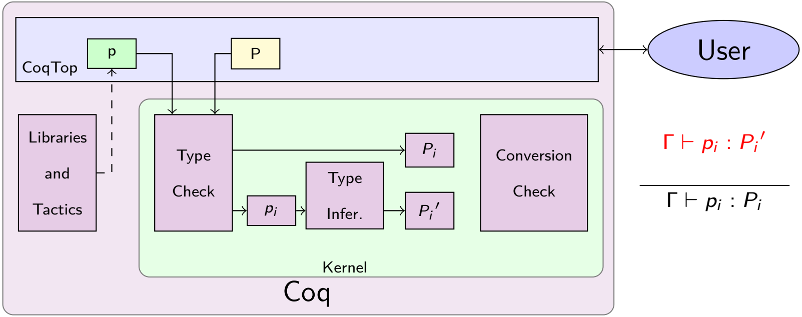


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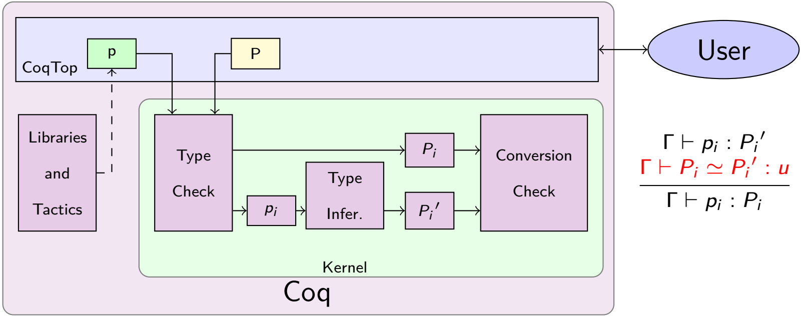


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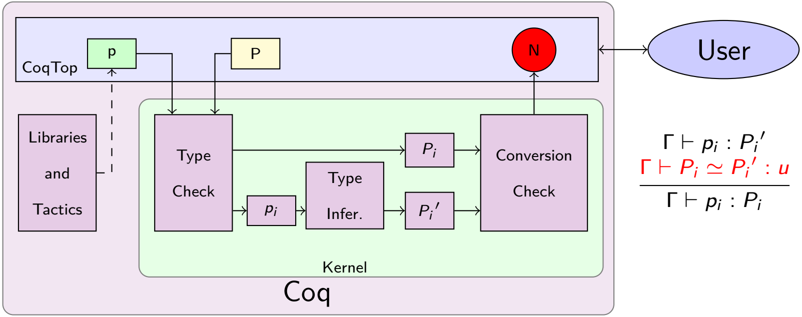


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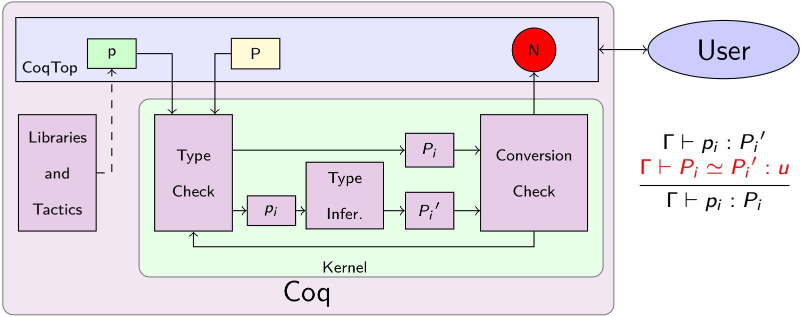


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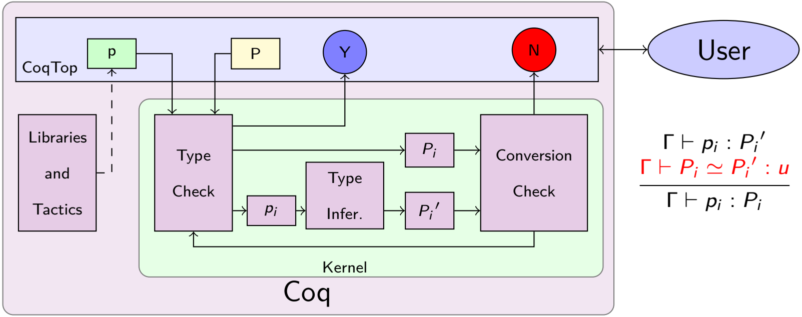


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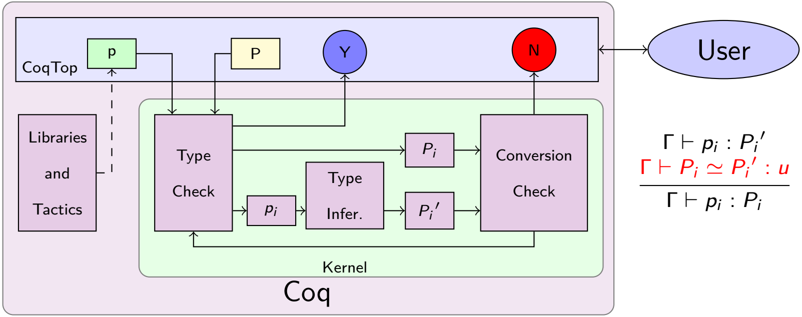


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- Conversion is purely intensional to ensure decidability.

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- The proof is carried out repeatedly at runtime.
- Equality generates explicit computations in proofs.

# Where does the problem originate from ?

- The kernel implements the Calculus of Inductive Constructions (CIC): the conversion rule carries out the conversion check:

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# Content

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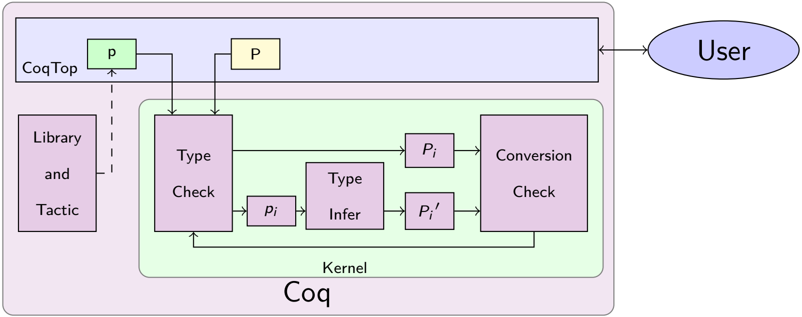

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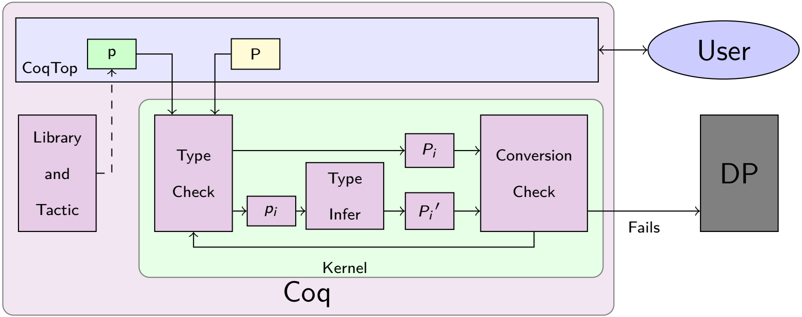

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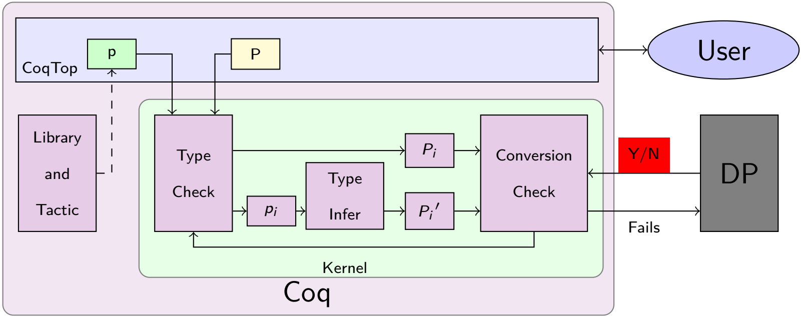

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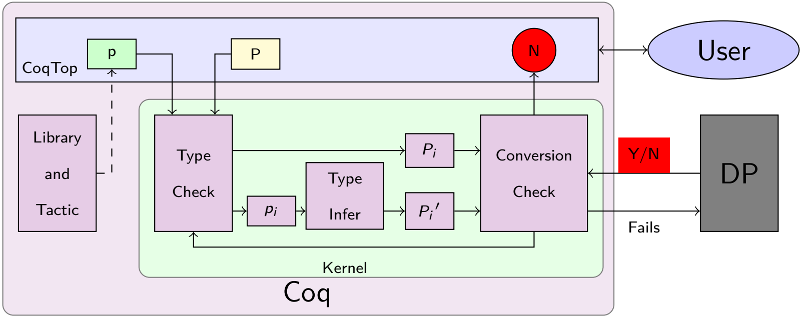

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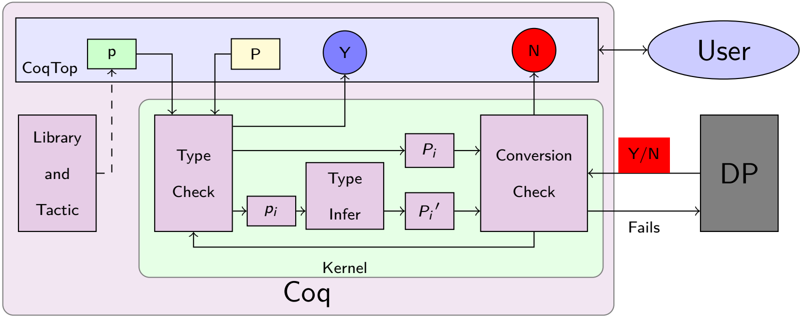

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$$\begin{tikzcd} m+(1+1)*n \arrow[d] \arrow[r, phantom, "\simeq"] & 2*n+(2-1)*m \arrow[d] \\ m+2n \arrow[r, phantom, "=_T"] & 2n+m \end{tikzcd}$$

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# Definition of $CIC^\omega(T)$

$CIC^\omega(T)$ contains $CIC^\omega$ and a $T$-inductive type $o$ of objects s.t.

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$$\frac{\Gamma \vdash t : \mathbf{nat} \quad \Gamma \vdash P : \forall[x : \mathbf{nat}].\,\{\mathbf{Prop}, \mathbf{Type}\} \qquad \Gamma \vdash f_{\mathbf{0}} : P\ \mathbf{0} \quad \Gamma \vdash f_{\mathbf{S}} : \forall[x : \mathbf{nat}].\,(P\ x \to P\ (\mathbf{S}\ x))}{\Gamma \vdash \textsc{Elim}_{nat}(P, f_{\mathbf{0}}, f_{\mathbf{S}}, t) : P\ t}$$

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- Elimination rules match their argument of type $o$ **modulo** $=_T$

# Definition of $CIC^{\omega}(T)$: Pseudo-Terms

$$
\begin{array}{rlr}
u, v, U, V ::= & \mathbf{Prop} \mid \mathbf{Type}_j & \text{(Universes)} \\
& \mid \mathcal{V} \mid u\ v \mid \lambda[x : U].\, v \mid \forall[x : U].\, V & \text{(CC)} \\
& \mid o \mid \mathcal{C} \mid \mathcal{D} \mid \textsc{Elim}_o(U, \overrightarrow{u}, v) & (T\text{-Inductives})
\end{array}
$$

## Reductions and Conversion

- $\beta$-reduction is defined as usual:

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- $\iota_T$-reduction generalizes pure $\iota$-reduction. For "Presburger"

  $$\mathrm{E{\scriptstyle LIM}}_{Nat}(P, f_{\mathbf{0}}, f_{\mathbf{S}}, v) \rightarrow \iota_T \begin{cases} f_{\mathbf{0}} & (1) \\ f_{\mathbf{S}}\ u\ \mathrm{E{\scriptstyle LIM}}_{Nat}(P, f_{\mathbf{0}}, f_{\mathbf{S}}, u) & (2) \end{cases}$$

  provided

  - $v =_T \mathbf{0}$ for case (1), and
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- The typing rules are as usual.

# Content

# Consistency and DTC proofs of fragments of $CIC^{\omega}(T)$

- CIC$^{\omega}$ : Paper proof of consistency and DTC
  B. Werner, "Sets in types, types in sets", in TACS : 1997
- $CIC^{1}(T)$: Implementation, paper proof of consistency and DTC,
  P.-Y. Strub, in CSL : 2010
- $CIC^{\omega}(T)$: Implementation, paper proof of consistency and DTC (restricted to weak-$T$-elimination)
  Barras, Jouannaud, Strub, Wang, "CoqMTU" in LICS : 2011
- $CIC^{\omega}(T)$: Formal proof of Consistency,
  Barras, Wang, in CSL : 2012
- $CIC^{\omega}(T)$: Paper proof of DTC,
  Jouannaud, Strub, in LPAR : 2017
- $CIC^{\omega}(T)$: Implementation, on-going.

# Strong normalization proof

$$\text{Let } \mathcal{T} = \{t \to C(\overline{u}) \ : \ C(\overline{u}) \text{ simplifies } t\}$$

**Lemma**

*$\mathcal{T}$ is a confluent and terminating rewriting system for $\leftrightarrow^*_T$.*

**Lemma**

$\longrightarrow_{\beta\iota_T} \subseteq \longrightarrow^+_{\beta\iota\mathcal{T}}$ *where* $\longrightarrow_{\beta\iota\mathcal{T}} \stackrel{\text{def}}{=} \longrightarrow_\beta \cup \longrightarrow_\iota \cup \longrightarrow_\mathcal{T}$.

We prove that $\longrightarrow_{\beta\iota\mathcal{T}}$ is SN by induction over $\longrightarrow_{\beta\iota} \cup \rhd$.

This proof uses syntactic arguments only, in particular the left-linearity of the rules in $\{\beta, \iota\}$ which provide with key commutation properties between $\{\beta, \iota\}$ and $\mathcal{T}$.

# Is strong elimination modulo needed in practice ? (1/2)

Assume we have a type constructor `poly : Type → Type` such that `poly K` stands for the type of polynomials in 1 indeterminate over `K`, we can construct the type `mpoly K n`, of multinomials over `n` indeterminates over `K` as:

```
Fixpoint mpoly (K : ring) (n : nat) : Type :=
match n with 0 => K | S p => poly (mpoly K p).
```

# Is elimination modulo needed in practice ? (2/2)

In the future version of CoqMT justified here, not only are `mpoly K (n+1+p)` and `mpoly K (p+n+1)` identified, which is not the case in Coq nor in the previous version of CoqMT, but because `(S (n+p))` simplifies `n+1+p` and `p+n+1`, they both compute to `poly (mpoly K (n+p))`, providing some canonical form of our initial type which highlights that `poly` is iterated at least once.

This would allow, for instance, to easily use properties on multivariate polynomials without relying on unnecessary type casts. Such needs arise quite naturally in the proof of the symmetric polynomials fundamental lemma, where all type casts occurring in the proof can be removed in CoqMT.

# Conclusion: when are casts needed ?

No type casts are ever needed in CoqMT provided the decidable theory $T$ contains the necessary syntax to express all equalities on dependent types whose proofs are needed to type the user's development.

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Casts become needed when the theory $T$ is undecidable.

# Last slide

Thank you for your attention